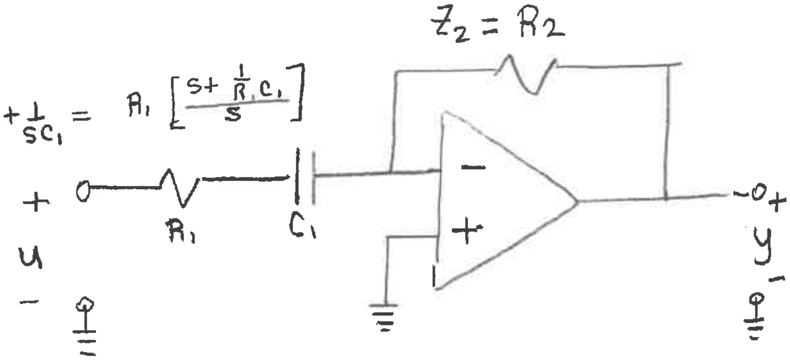
$\boxed{\text{Example 31}}$ (HPF, LPF, BPF Via Op-Amp)

$\boxed{2040}$

$\boxed{1}$ $\boxed{\text{High Pass Filter (HPF)}}$

$Z_1 = R_1 + \frac{1}{sC_1} = R_1\left[\frac{s+\frac{1}{R_1C_1}}{s}\right]$



Transfer function from u to y

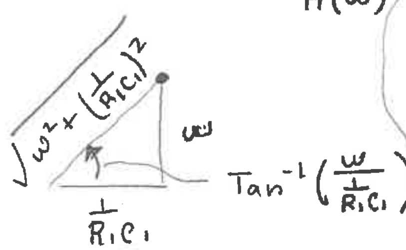
$$H = \frac{y}{u} = -\left(\frac{Z_2}{Z_1}\right) = -\left[\frac{R_2}{R_1\left[\frac{s+\frac{1}{R_1C_1}}{s}\right]}\right] \Rightarrow \boxed{H(s) = -\left(\frac{R_2}{R_1}\right)\left[\frac{s}{s+\frac{1}{R_1C_1}}\right]}$$

$H(\infty)$

Note:
$H \cong -R_2C_1 s$ for small s
$H \cong -\frac{R_2}{R_1}$ for large s

$jw = we^{j90°}$

$$H(jw) = -\left(\frac{R_2}{R_1}\right)\left[\frac{jw}{jw+\frac{1}{R_1C_1}}\right]$$

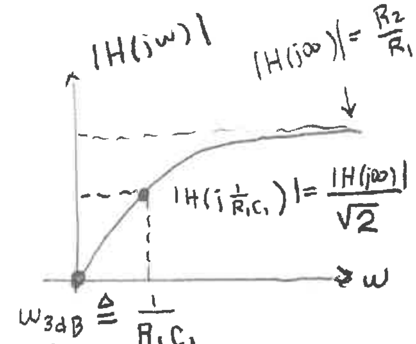


$$= \left(\frac{R_2}{R_1}\right)e^{j180°}\left[\frac{we^{j90°}}{\sqrt{w^2+\left(\frac{1}{R_1C_1}\right)^2}\; e^{j\tan^{-1}\left(\frac{w}{\frac{1}{R_1C_1}}\right)}}\right]$$

$$= \frac{\left(\frac{R_2}{R_1}\right) w}{\sqrt{w^2+\left(\frac{1}{R_1C_1}\right)^2}}\; e^{j\left(180+90-\tan^{-1}\left(\frac{w}{\frac{1}{R_1C_1}}\right)\right)}$$

Magnitude Response $|H(jw)| = \left(\frac{R_2}{R_1}\right)\left[\frac{w}{\sqrt{w^2+\left(\frac{1}{R_1C_1}\right)^2}}\right]$

$|H(j\infty)|$

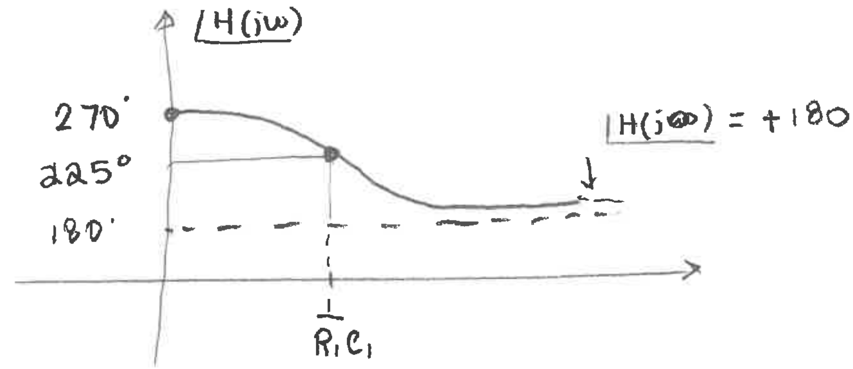


H is a High Pass Filter (HPF)

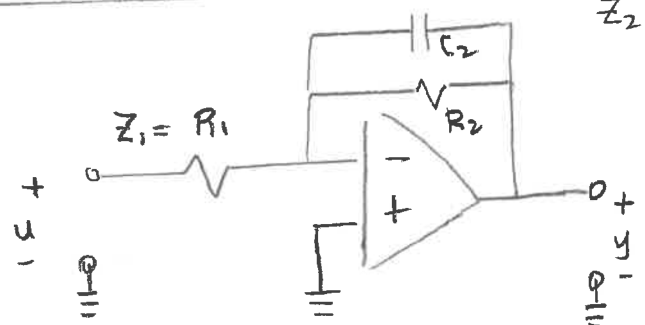
Example 31

$$\angle H(j\omega) = 270^\circ - \tan^{-1}\left(\frac{\omega}{\frac{1}{R_1C_1}}\right)$$

Phase Response



2 **Low Pass Filter (LPF)**



$$Z_2 = R_2 \| \frac{1}{sC_2} = \frac{\frac{R_2}{sC_2}}{R_2 + \frac{1}{sC_2}} = \frac{\frac{1}{C_2}}{s + \frac{1}{R_2C_2}}$$

Transfer function from u to y

$$H = \frac{y}{u} = -\left(\frac{Z_2}{Z_1}\right) = -\frac{\left[\frac{\frac{1}{C_2}}{s+\frac{1}{R_2C_2}}\right]}{R_1} \implies \boxed{H(s) = -\left(\frac{R_2}{R_1}\right)\left[\frac{\frac{1}{R_2C_2}}{s+\frac{1}{R_2C_2}}\right]}$$

$-H(0)$

Note:
$H(s) \cong -\left(\frac{R_2}{R_1}\right)$ for s small

$H(s) \cong -\frac{\frac{1}{R_1C_2}}{s}$ for s large

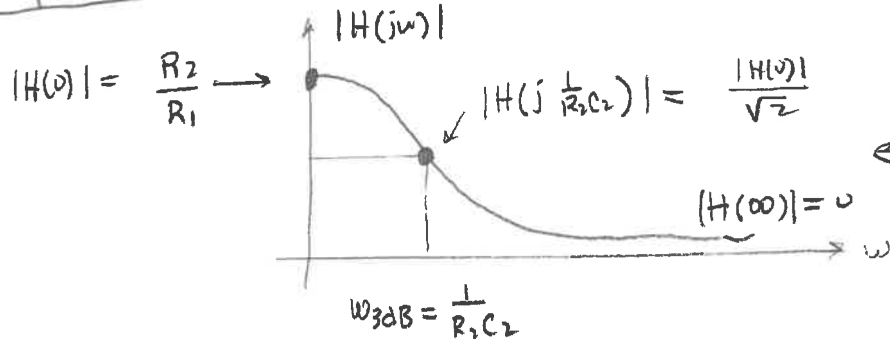
$$H(j\omega) = -\left(\frac{R_2}{R_1}\right)\left[\frac{\frac{1}{R_2C_2}}{j\omega + \frac{1}{R_2C_2}}\right]$$

$$= \frac{\left(\frac{R_2}{R_1}\right)e^{j180^\circ}\left(\frac{1}{R_2C_2}\right)e^{j0^\circ}}{\sqrt{\omega^2 + \left(\frac{1}{R_2C_2}\right)^2}\; e^{j\tan^{-1}\left(\frac{\omega}{\frac{1}{R_2C_2}}\right)}}$$

Example 31

$$|H(j\omega)| = \underbrace{\left(\frac{R_2}{R_1}\right)}_{|H(0)|} \left[\frac{\frac{1}{R_2C_2}}{\sqrt{\omega^2 + \left(\frac{1}{R_2C_2}\right)^2}} \right]$$

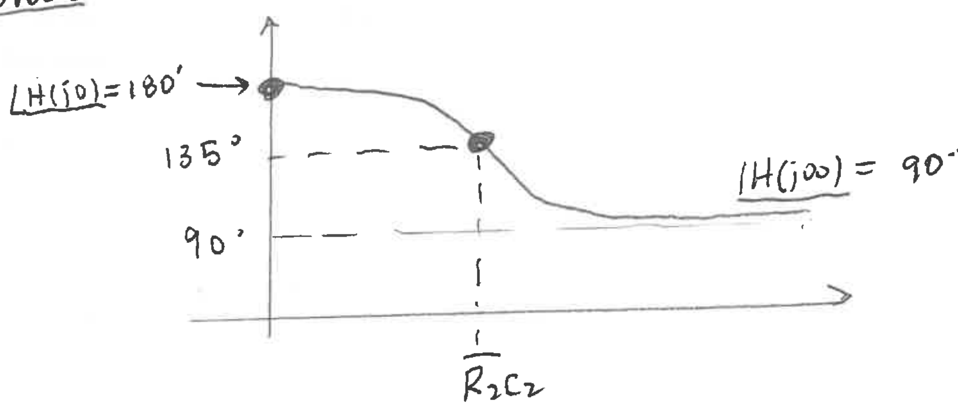
Magnitude Response



← H is a Low Pass Filter (LPF)

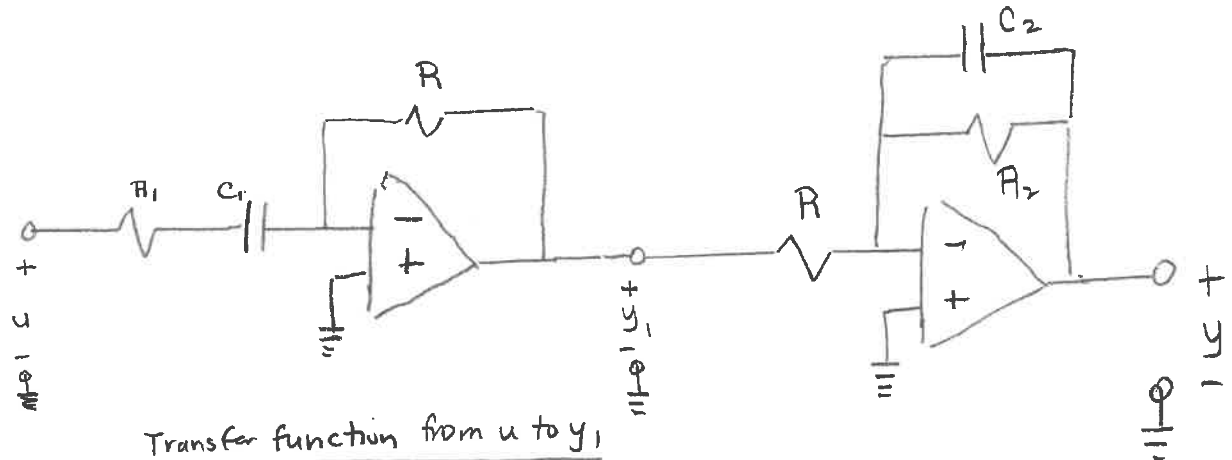
Phase Response

$$\angle H(j\omega) = 180° - \tan^{-1}\left(\frac{\omega}{\frac{1}{R_2C_2}}\right)$$



3 Band Pass Filter (BPF)

– HPF followed by a LPF



Transfer function from u to y_1

$$y_1 = -\left(\frac{R}{R_1}\right)\left[\frac{s}{s+\frac{1}{R_1C_1}}\right]u$$

Assume: $\boxed{\frac{1}{R_1C_1} << \frac{1}{R_2C_2}}$

$$y_2 = -\left(\frac{R_2}{R}\right)\left[\frac{\frac{1}{R_2C_2}}{s+\frac{1}{R_2C_2}}\right]y_1 = \left(\frac{R_2}{R}\right)\left[\frac{\frac{1}{R_2C_2}}{s+\frac{1}{R_2C_2}}\right]\left(\frac{R}{R_1}\right)\left[\frac{s}{s+\frac{1}{R_1C_1}}\right]u$$

Transfer function from y_1 to y_2

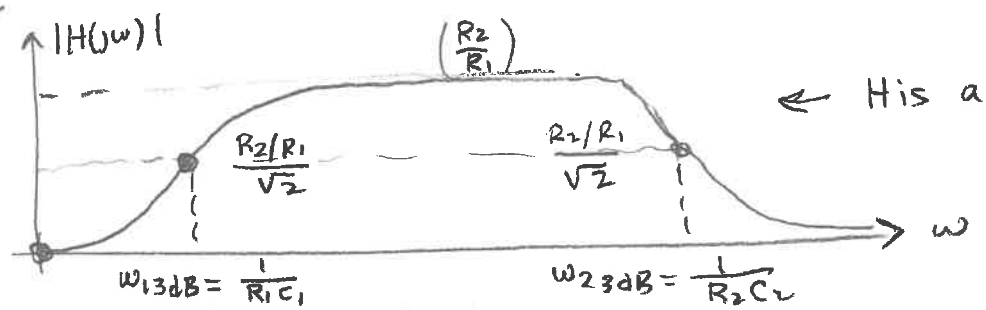
$$H(s) = \left[\frac{s}{s+\frac{1}{R_1C_1}}\right]\left(\frac{R_2}{R_1}\right)\left[\frac{\frac{1}{R_2C_2}}{s+\frac{1}{R_2C_2}}\right]$$

Transfer function from u to y

$$H(j\omega) = \left[\frac{j\omega}{j\omega+\frac{1}{R_1C_1}}\right]\left(\frac{R_2}{R_1}\right)\left[\frac{\frac{1}{R_2C_2}}{j\omega+\frac{1}{R_2C_2}}\right]$$

$$|H(j\omega)| = \left[\frac{\omega}{\sqrt{\omega^2+\left(\frac{1}{R_1C_1}\right)^2}}\right]\left(\frac{R_2}{R_1}\right)\left[\frac{\frac{1}{R_2C_2}}{\sqrt{\omega^2+\left(\frac{1}{R_2C_2}\right)^2}}\right]$$

Magnitude Response



← H is a Band Pass Filter (BPF)

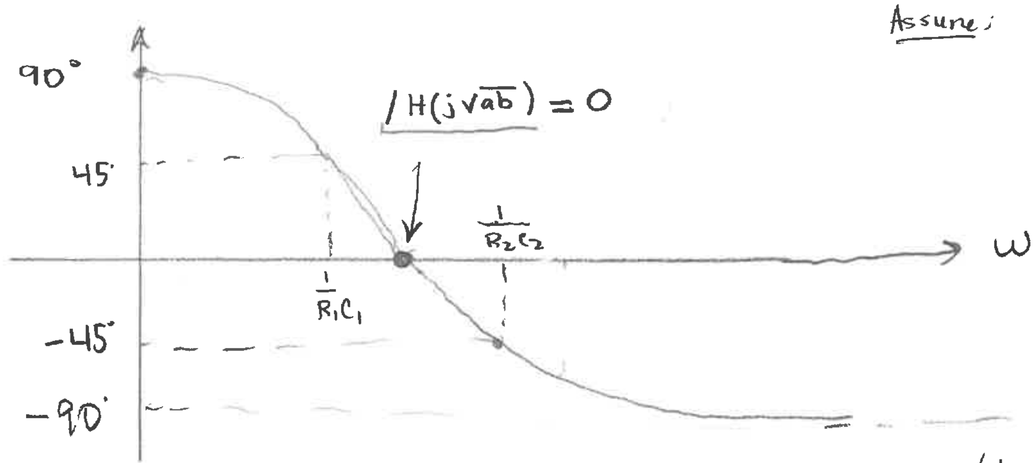
Example 31

Phase Response

$$\angle H(j\omega) = 90^\circ - \operatorname{Tan}^{-1}\left(\frac{\omega}{\underbrace{\frac{1}{R_1C_1}}_{a}}\right) - \operatorname{Tan}^{-1}\left(\frac{\omega}{\underbrace{\frac{1}{R_2C_2}}_{b}}\right)$$

Assume: $\frac{1}{R_1C_1} << \frac{1}{R_2C_2}$

$\angle H(j0) = 90^\circ$



$\angle H(j\infty) = -90^\circ$

Lets show $\angle H(j\sqrt{ab}) = 0$.

Let $\theta(\omega) = \operatorname{Tan}^{-1}\left(\frac{\omega}{a}\right) + \operatorname{Tan}^{-1}\left(\frac{\omega}{b}\right)$

$$\operatorname{Tan}[\theta(\omega)] = \frac{\frac{\omega}{a} + \frac{\omega}{b}}{1 - \left(\frac{\omega^2}{ab}\right)} \Rightarrow$$

$$\boxed{\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

$$\operatorname{Tan}[\theta(\sqrt{ab})] = \frac{\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}}{1 - 1} = \frac{\quad}{0}$$
$$= \infty$$

$\Rightarrow \theta(\sqrt{ab}) = 90^\circ$

$\Rightarrow \angle H(j\sqrt{ab}) = 90 - 90^\circ$
$= 0^\circ$

Example 31

<u>Note:</u> The "Bode analyzer" software in the lab is desigred to plot the

magnitude response $|H(j\omega)|$

& phase response $\angle H(j\omega)$

versus frequency ω.

Together, they are the frequency response of the system H

Why are these important?

$|H(j\omega)|$ & $\angle H(j\omega)$

are important because of the following EXTREMELY IMPORTANT RESULT:

input $\quad$ output

$$u \to \boxed{H} \to y$$

stable (all poles in LHP)
i.e. Re pole < 0

If

$$u(t) = A + B \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta)$$

then

$$y_{ss} = A\,H(0) + B\,|H(j\omega_0)| \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta + \angle H(j\omega_0))$$

steady state output

Problem 31

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For each of the following

a) compute $|H(j\omega)|$, $\angle H(j\omega)$

b) plot $|H(j\omega)|$ & $\angle H(j\omega)$ vs ω

c) show how to construct H using op-amps & resistors.

$\boxed{1}$ $H(s) = 100\left[\frac{s}{s+1}\right]$

$\boxed{2}$ $H(s) = 100\left[\frac{10^3}{s+10^3}\right]$

$\boxed{3}$ $H(s) = \left[\frac{s}{s+1}\right](100)\left[\frac{10^3}{s+10^3}\right]$