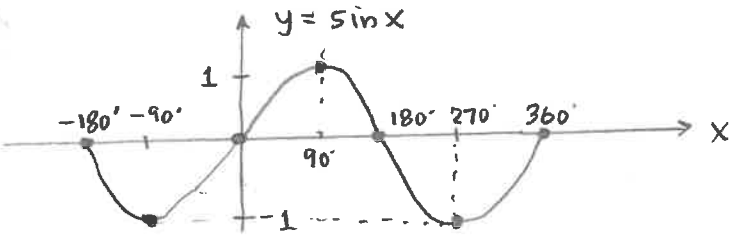
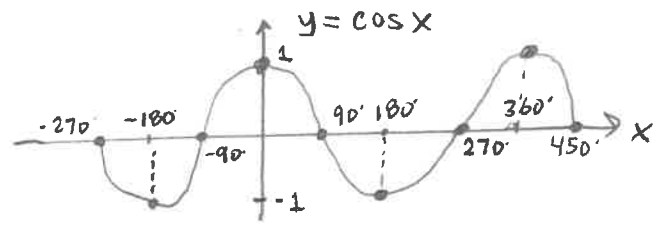
$\boxed{\text{Example 27}}$ (Review of Some Trigonometry) $\boxed{1430}$

Mon 3/18/2'





$\boxed{\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \end{aligned}}$ ← odd function / ← even function

Some useful Trig Identities

$$\boxed{\sin(x \mp 90^\circ) = \mp \cos x}$$

when sin is shifted 90° to right, you get −cos

when sin " " " " left, " " +cos

$$\boxed{\cos(x \mp 90^\circ) = \pm \sin x}$$

when cos is shifted 90° to right, you get +sin

" " " " " left, " " −sin

$$\boxed{\sin(x \mp 180^\circ) = -\sin x}$$

when sin is shifted 180° left or right, you get −sin

$$\boxed{\cos(x \mp 180^\circ) = -\cos x}$$

when cos " " " " " " , you get −cos

Note: I do NOT memorize the above identities! I know the sin & cos wave forms & I understand how the above shifting affects the wave form!

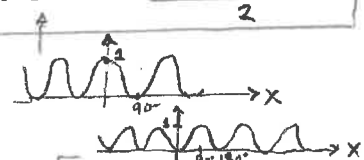
Remember!

$$\boxed{\cos x \cos y = \tfrac{1}{2}\cos(\underset{\text{sum}}{x+y}) + \tfrac{1}{2}\cos(\underset{\text{difference}}{x-y})}$$

← This is TATOOED IN MY BRAIN!

Sum-Difference cos formula

$$\Rightarrow \cos x \cos x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\underbrace{\cos(0^\circ)}_{1} \Longrightarrow \boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$



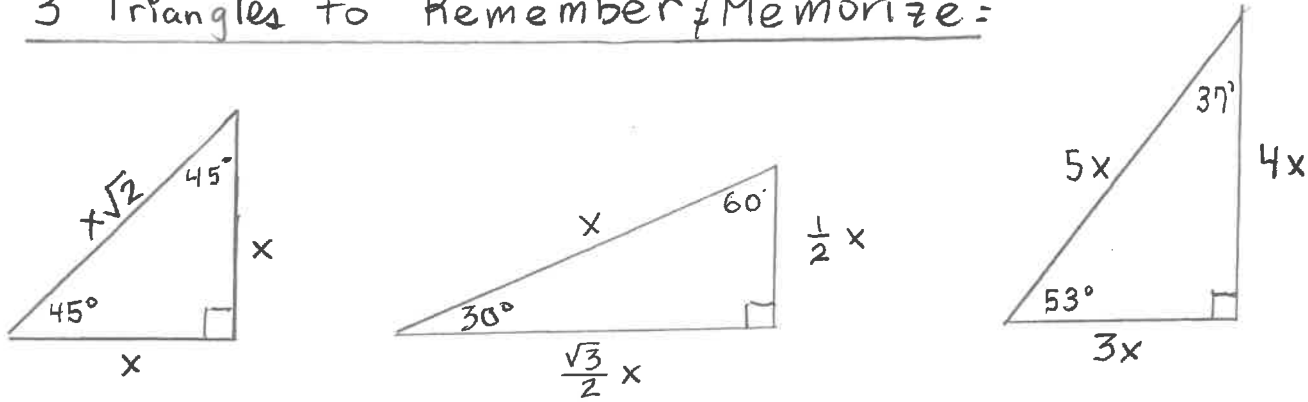
$$\sin^2 x = [\cos(x-90)]^2$$

$$= \tfrac{1}{2}\cos(2x-180^\circ) + \tfrac{1}{2}\underbrace{\cos(0^\circ)}_{1}$$

$$= -\tfrac{1}{2}\cos 2x + \tfrac{1}{2} \Longrightarrow \boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$$

Example 27 (Review of Some Trigonometry)

3 Triangles to Remember/Memorize:

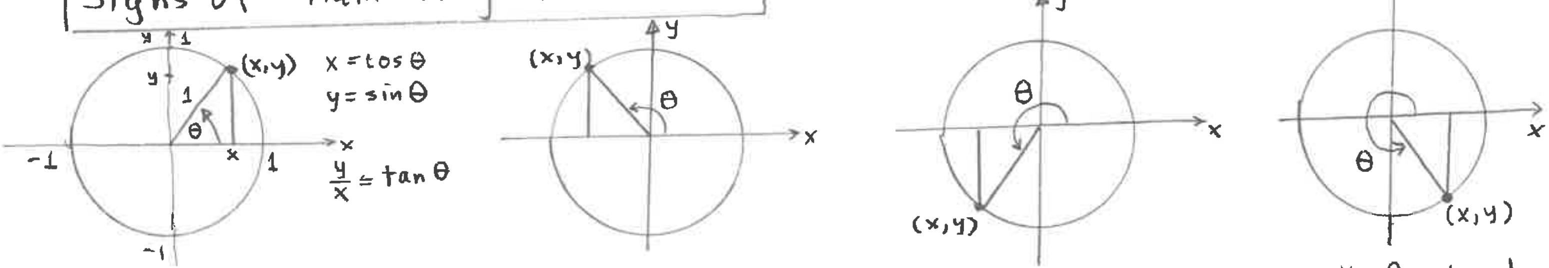


Note: There is nothing "sacred" about these triangles; i.e. it is not like they naturally arise in many real-world applications --- they do not!

Why should they be remembered/memorized?

Because knowing their well-known simple trigonometry will permit you to rapidly solve problems with nicely "cooked-up" numbers so that you can more easily master the associated important fundamental concepts!

Signs of Main Trig Functions



1st Quadrant	2nd Quadrant	3rd Quadrant	4th Quadrant
sin, cos, tan > 0	sin > 0, cos < 0, tan < 0	sin, cos < 0, tan > 0	sin < 0, cos > 0, tan < 0

mnemonic:

ALL	Students	Take	Calculus
all > 0 in 1st quad	sin > 0 in 2nd quad	tan > 0 in 3rd quad	cos > 0 in 4th quad

Example 27

$\boxed{\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)}$

Application of Sum-Difference Cos Formula to Power & Average Power

1450

$i(t) = |I| \cos(\omega_i t + \theta_i)$

$+$

$v(t) = |V| \cos(\omega_v t + \theta_v)$

$-$

equal by definition

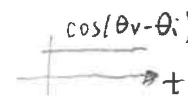
$$p(t) \stackrel{\Delta}{=} v(t)\, i(t)$$

$$= |V| \cos(\omega_v t + \theta_v)\, |I| \cos(\omega_i t + \theta_i)$$

$$= \frac{|V||I|}{2} \cos\big(\underbrace{(\omega_v+\omega_i)}_{\text{sum of frequencies}} t + \theta_v + \theta_i\big) + \frac{|V||I|}{2} \cos\big(\underbrace{(\omega_v-\omega_i)}_{\text{Difference of frequencies}} t + (\theta_v - \theta_i)\big)$$

Now let $\omega_v = \omega_i = \omega$

$$\Longrightarrow \quad p(t) = \frac{|V||I|}{2} \Big[\underbrace{\cos(2\omega t + \theta_v + \theta_i)}_{\text{double frequency term}} + \underbrace{\cos(\theta_v - \theta_i)}_{\text{constant term}} \Big]$$



If we now take the average of both sides

i.e. $\langle f \rangle = \frac{1}{T} \int_T f(t)\, dt$

integral over any period for the periodic function f

$$\Longrightarrow \quad \langle p \rangle = \text{average of double freq term} + \text{average of constant}$$

$$= 0 + \frac{|V||I|}{2} \cos(\theta_v - \theta_i)$$

Average Power for Sinusoidal v & i

Summary

$i(t) = |I| \cos(\omega t + \theta_i)$

$+$

$v(t) = |V| \cos(\omega t + \theta_v)$

$-$

$$\langle p \rangle = \frac{|V||I|}{2} \cos(\theta_v - \theta_i)$$

$$\underbrace{p(t)}_{\text{instantaneous power}} = v(t)\, i(t) = \frac{|V||I|}{2}\Big[\underbrace{\cos(2\omega t + \theta_v + \theta_i)}_{\text{double frequency term}} + \underbrace{\cos(\theta_v - \theta_i)}_{\text{constant term}}\Big]$$

Problem 27 (Average Power for R, L, C)

Consider a device

$i = |I| \cos(\omega t + \theta_i)$

$+$

$v(t) = |V| \cos(\omega t + \theta_v)$

$-$

It has been shown that

$$\langle p \rangle = \frac{|V||I|}{2} \cos(\theta_v - \theta_i) \qquad \text{pg } 1450$$

a) Suppose device is a resistor $(v = Ri)$

Show that

$$\boxed{\begin{array}{l} \theta_v - \theta_i = 0 \quad \therefore \quad \cos(\theta_v - \theta_i) = 1 \\ \langle p \rangle_R = \dfrac{|V||I|}{2} = \dfrac{R|I|^2}{2} = \dfrac{|V|^2}{2R} \end{array}}$$

b) Suppose device is a capacitor $\left(i = C\frac{dv}{dt}\right)$

$\left(I = sCV \Rightarrow \frac{V}{I} = \frac{1}{sC} \Rightarrow \frac{V}{I} = \frac{1}{j\omega C}\right)$

voltage lags current by 90°

$$\boxed{\begin{array}{l} \theta_v - \theta_i = -90^\circ \\ \cos(\theta_v - \theta_i) = 0 \\ \langle p_C \rangle = 0 \end{array}}$$

c) Suppose device is an inductor $\left(v = L\frac{di}{dt}\right)$

voltage leads current by 90°

$\left(V = (sL)I \Rightarrow \frac{V}{I} = sL, \quad \frac{V}{I} = j\omega L\right)$

$$\boxed{\begin{array}{l} \theta_v - \theta_i = +90 \\ \cos(\theta_v - \theta_i) = 0 \\ \langle p_L \rangle = 0 \end{array}}$$

Example 28

Mastering (Complex Numbers !) 1470

Note: Complex #'s were introduced by Cardano in 1545 while he was finding the roots of a cubic.

1st Advice: Draw lots of pictures !

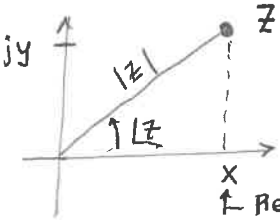
This is critical to mastering complex #'s & using them proficiently when analyzing dynamical systems.

Notation:

We will write complex #'s as =

often used by mathematicians → $z = x + jy$ (rectangular form)

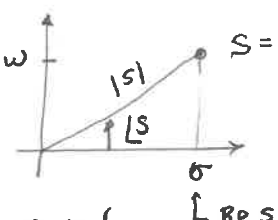
$j\,\mathrm{Im}\,z \rightarrow jy$



polar form: $z = |z|e^{j\angle z}$ complex z-plane

often used by engineers → $s = \sigma + j\omega$

$j\,\mathrm{Im}\,s \rightarrow j\omega$



polar form: $s = |s|e^{j\angle s}$ complex s-plane

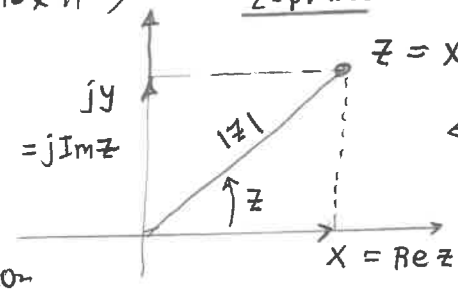
Note: Can & will be associated with exponential sinusoid $A e^{-\sigma t}\cos(\omega t + \theta)$

Lets start with some basics.

Example 28

(Mastering Complex #'s)

complex z-plane



← this picture shows a 1st quadrant angle

Consider the complex # = $\boxed{j \stackrel{\Delta}{=} \sqrt{-1}}$

$$z = x + jy$$

rectangular form of z

equals by definition

always true

$$\mathrm{Re}\, z \stackrel{\Delta}{=} x = |z| \cos \angle z$$

$$\mathrm{Im}\, z \stackrel{\Delta}{=} y = |z| \sin \angle z$$

$$|z| = \sqrt{(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2} = \sqrt{x^2 + y^2}$$

from picture & Pythagorean Theorem

Note:

$\mathrm{Im}\, z$ is NOT jy

don't EVER make this mistake!

Convention: Always measured with respect to positive "x-axis"

NOT always true!

$$\angle z = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) \leftarrow$$

from 1st quadrant angle picture shown & basic trig

Note:

We will see below that this does NOT hold for any sign of x & y!

It is true (i.e. $\angle z = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$) for

- 1st quad angles $(x, y > 0)$
- 4th quad angles $(x > 0, y < 0)$

It is not true for

- 2nd quad angles $(x < 0, y > 0)$
- 3rd quad angles $(x, y < 0)$

always true

$$z = |z|\, e^{j\angle z}$$

polar form of z

... here we go on our complex # journey ...

.... we begin by examining complex #'s in each quadrant.

Example 28

(Mastering Complex #'s)

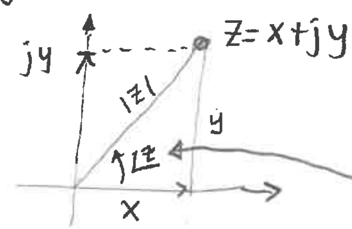
This slide show the importance of pictures... for understanding $\angle z$!

1st Quadrant

$(x, y > 0)$

$z = x + jy$

↑ rectangular form



$\text{Re}\, z = x = |z| \cos \angle z$

$\text{Im}\, z = y = |z| \sin \angle z$

$|z| = \sqrt{x^2 + y^2}$

$$\boxed{\angle z = \text{Tan}^{-1}\left(\frac{y}{x}\right)}$$

$z = |z| e^{j\angle z}$

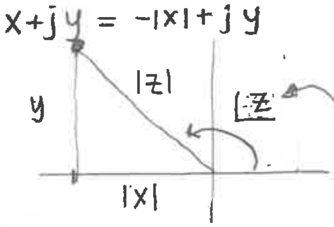
$= \sqrt{x^2+y^2}\, e^{j\left(\text{Tan}^{-1}\left(\frac{y}{x}\right)\right)}$

↑ polar form

2nd Quadrant

$(x < 0, y > 0)$

$z = x + jy = -|x| + jy$



$\text{Re}\, z = x = -|x| = |z| \cos \angle z$

$\text{Im}\, z = y = |z| \sin \angle z$

$|z| = \sqrt{x^2+y^2}$

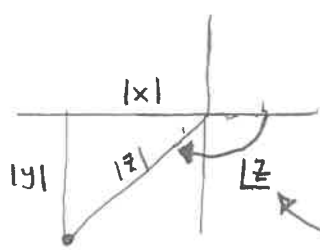
$$\boxed{\angle z = 180 - \text{Tan}^{-1}\left(\frac{y}{|x|}\right)}$$

$z = |z| e^{j\angle z}$

$= \sqrt{x^2+y^2}\, e^{j\left(180^\circ - \text{Tan}^{-1}\left(\frac{y}{|x|}\right)\right)}$

3rd Quadrant

$(x, y < 0)$



(convention: use the "smaller" angle)

$z = x + jy = -|x| - j|y|$

$\text{Re}\, z = x = -|x| = |z| \cos \angle z$

$\text{Im}\, z = y = -|y| = |z| \sin \angle z$

$|z| = \sqrt{x^2+y^2}$

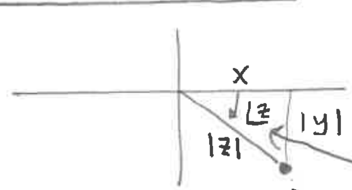
$$\boxed{\angle z = -180^\circ + \text{Tan}^{-1}\left(\frac{|y|}{|x|}\right)}$$

$z = |z| e^{j\angle z}$

$= \sqrt{x^2+y^2}\, e^{j\left(-180^\circ + \text{Tan}^{-1}\left(\frac{|y|}{|x|}\right)\right)}$

4th Quadrant

$(x > 0, y < 0)$



$z = x + jy = x - j|y|$

$\text{Re}\, z = x$

$\text{Im}\, z = y = -|y|$

$|z| = \sqrt{x^2+y^2}$

$$\boxed{\angle z = -\text{Tan}^{-1}\left(\frac{|y|}{x}\right)}$$

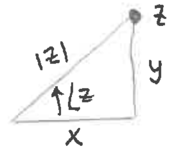
$z = |z| e^{j\angle z} = \sqrt{x^2+y^2}\, e^{-j\, \text{Tan}^{-1}\left(\frac{|y|}{x}\right)}$

Note: $\boxed{\angle z = \text{Tan}^{-1}\left(\frac{y}{x}\right) \text{ only for 1st \& 4th quadrant angles !}}$

Summary of Results That Always Hold

(for any x, y)

Let $z = x + jy$



← drawn for $x, y > 0$ (1st quadrant angle)

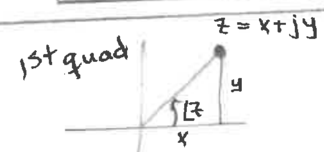
$$\operatorname{Re} z = |z| \cos \angle z = x$$

$$\operatorname{Im} z = |z| \sin \angle z = y$$

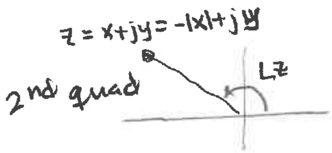
$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{x^2 + y^2}$$

$$z = x + jy = |z| \cos \angle z + j\,|z| \sin \angle z = |z| e^{j\angle z}$$

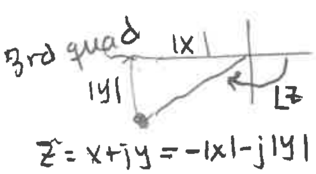
For:



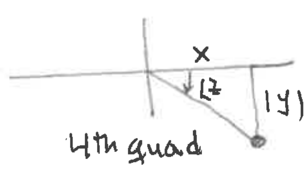
$$\angle z = \operatorname{Tan}^{-1}\left(\frac{y}{x}\right)$$



$$\angle z = 180 - \operatorname{Tan}^{-1}\left(\frac{y}{|x|}\right)$$



$$\angle z = -180 + \operatorname{Tan}^{-1}\left(\frac{|y|}{|x|}\right)$$



$$\angle z = -\operatorname{Tan}^{-1}\left(\frac{|y|}{x}\right)$$

Warning: Draw pictures to get the angles right!

Recall:

$$\boxed{\bar{z} \triangleq |z| e^{-j\angle z} = \operatorname{Re} z - j \operatorname{Im} z = x - jy}$$

conjugate of z (for $\triangleq$) — can be shown with some arithmetic (for $=$)

Example 28 (Mastering complex #'s)

Arithmetic Operations:

Let $z_1 = x_1 + jy_1 = |z_1| e^{j\angle z_1}$

$z_2 = x_2 + jy_2 = |z_2| e^{j\angle z_2}$

Addition

$$\boxed{\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2) \end{aligned}}$$

Note: Rectangular forms facilitate addition!

∴ .. just need to

add real parts to get new real part
" imag " " " " imag "

Using polar forms:

$z_1 + z_2 = |z_1|e^{j\angle z_1} + |z_2| e^{j\angle z_2}$

$= (|z_1|\cos\angle z_1 + j|z_1|\sin\angle z_1) + (|z_2|\cos\angle z_2 + j|z_2|\sin\angle z_2)$

$= (x_1 + jy_1) + (x_2 + jy_2)$

$= (x_1 + x_2) + j(y_1 + y_2)$

using polar forms takes more work for addition!

Multiplication

EASY! angles add

$$\boxed{z_1 z_2 = (|z_1| e^{j\angle z_1})(|z_2| e^{j\angle z_2}) = |z_1||z_2| e^{j(\angle z_1 + \angle z_2)}}$$

magnitudes multiply

Here, the polar forms are the way to go!

Using rectangular forms:

$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$

$= (x_1)(x_2 + jy_2) + (jy_1)(x_2 + jy_2)$

$= x_1 x_2 + jx_1 y_2 + jy_1 x_2 + (j)(j) y_1 y_2$ $\quad \boxed{j^2 \triangleq -1}$

$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$ ← (just gathered real & imag parts)

Use distributive property of multiplication over addition!

This is much more work!

It can be shown (after alot of algebra) that

$(x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) = |z_1||z_2| e^{j(\angle z_1 + \angle z_2)}$

How about division - - - -

Example 28 (Mastering complex #5)

1520

Division

angle (top) - angle (bottom)

EASY!

$$\frac{Z_1}{Z_2} = \frac{|Z_1|\, e^{j\angle Z_1}}{|Z_2|\, e^{j\angle Z_2}} = \frac{|Z_1|}{|Z_2|}\, e^{j(\angle Z_1 - \angle Z_2)}$$

(magnitude of top) / (magnitude of bottom)

Here, the polar forms are the way to go!

remember this trick

Using rectangular forms:

$$\frac{Z_1}{Z_2} = \left(\frac{x_1+jy_1}{x_2+jy_2}\right) = \left(\frac{x_1+jy_1}{x_2+jy_2}\right)\left(\frac{x_2-jy_2}{x_2-jy_2}\right)$$

$$= \frac{x_1x_2 - jx_1y_2 + jy_1x_2 - j^2 y_1y_2}{x_2^2 - j\cancel{x_2y_2} + j\cancel{y_2x_2} - j^2 y_2^2} \qquad j^2 \triangleq -1$$

$$= \frac{(x_1x_2 + y_1y_2) + j(-x_1y_2 + x_2y_1)}{x_2^2+y_2^2}$$

$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + j\left(\frac{-x_1y_2+x_2y_1}{x_2^2+y_2^2}\right)$$

This is much more work!

after alot of algebra one can show that this

$$= \frac{|Z_1|}{|Z_2|}\, e^{j(\angle Z_1 - \angle Z_2)}$$

Example 28 (Mastering complex #'s)

Here are

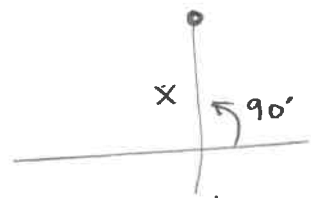
Some simple complex #'s: (assume $x > 0$)

$x = x e^{j0^\circ}$

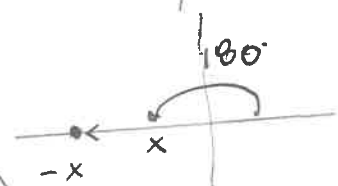
Angle of a positive # is 0° ($\pm$ any integer multiple of 360° ... but it is convention to take 0° as best answer!)

> Magnitude of a positive # is simply the #!
> Angle of a positive # is 0°

$jx = x e^{j90^\circ}$



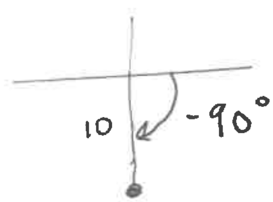
$-x = x e^{j180^\circ}$



— (angle here can also be taken as -180° or $180 \pm$ any integer multiple of 360° ... but it is convention to take 180° as best answer!)

> Magnitude of a negative # is simply $|\#|$ (absolute value of #)
> Angle of a negative # is 180°

$-jx = x e^{-j90^\circ}$



Message on Approximation:

$$\boxed{z + \Delta \cong z \quad \text{when} \quad |\Delta| < \tfrac{1}{10}|z|}$$

e.g. $j1 + 0.1 \cong j1 = 1 e^{j90^\circ}$

$10 \pm j1 \cong 10 = 10 e^{j0^\circ}$

$-j1 + 0.1 \cong -j1 = 1 e^{-j90^\circ}$

$-100 \pm j10 \cong -100 = 100 e^{j180^\circ}$

Example 28 (Mastering Complex #5)

lets take ~~our~~ complex # pictures for a ride

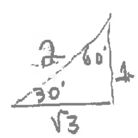
... to see how they can be used in practice

(i.e. analysis of dynamical systems)

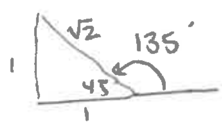
$$Z = \frac{(\sqrt{3} + j1)(-1 + j1)(-4 - j3)(100 + j1)}{(j10)(6 - j8)(0.1 - j1)(-9 + j10)}$$

<u>we want to Find $|Z|$, $\angle Z$, $\mathrm{Re}\,Z$, $\mathrm{Im}\,Z$</u> ... (approximate when it makes sense to do so!)

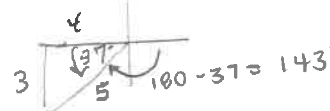
Lets begin by drawing pictures:



$\sqrt{3} + j1 = 2e^{j30°}$

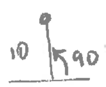


$-1 + j1 = \sqrt{2}\, e^{j135°}$

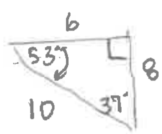


$-4 - j3 = 5e^{-j143°}$

$100 + j1 \cong 100e^{j0°}$

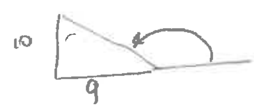


$j10 = 10e^{j90°}$



$6 - j8 = 10e^{-j53°}$

$0.1 - j1 \cong -j1$
$= 1e^{-j90°}$



$|-9 + j10|$
$= \sqrt{81 + 100}$ (near $10\sqrt{2}$)

$\angle -9 + j10$
$= 180° - \mathrm{Tan}^{-1}\left(\frac{10}{9}\right)$
(near 135°)

approximate because we have approximated!

$$|Z| = \frac{|\text{top}|}{|\text{bottom}|} \cong \frac{(2)(\sqrt{2})(5)(\overset{\text{approx}}{100})}{(\underset{\text{approx}}{10})(10)(\underset{\text{approx}}{1})(\sqrt{81+100})}$$

$$\angle Z = \angle \text{top} - \angle \text{bottom} = \left[(30°) + (135°) + (-143°) + (0°)\right]$$
$$- \left[(90°) + (-53°) + (-90°) + \left(180° - \mathrm{Tan}^{-1}\left(\tfrac{10}{9}\right)\right)\right]$$

Example 28 (Mastering Complex #'s) 1550

Consider

$$H(s) = \frac{(s+1)\,(s-2)\,(3-s)\,(s^2+2s+2)}{s^2\,(s^2+s+1)\,(s^2-8s+25)}$$

Compute $|H(j\omega)|$, $\angle H(j\omega)$ for each ω below

(approximate when it makes sense to do so)

$\boxed{\omega = 0.1}$

$$H(j0.1) \cong \frac{(\underbrace{0.+1}_{1e^{j0^\circ}})\,(\underbrace{0-2}_{2e^{j180^\circ}})\,(\underbrace{3-0}_{3e^{j0^\circ}})\,(\underbrace{0+0+2}_{2e^{j0^\circ}})}{(1e^{j90^\circ})^2\,(\underbrace{0+0+1}_{1e^{j0^\circ}})\,(\underbrace{0-0+25}_{25e^{j0^\circ}})}$$

$$\cong \underbrace{\frac{(1)(2)(3)(2)}{(1^2)(1)(25)}}_{\sim|H(j0.1)|}\; e^{j\overbrace{(180^\circ-2(90^\circ))}^{\sim\angle H(j0.1)}}$$

$\boxed{\omega = 1}$

$$H(j1) = \frac{(j1+1)\,(j1-2)\,(3-j1)\,(-1+j2+2)}{(1e^{j90^\circ})^2\,(-1+j1+1)\,(-1-j8+25)}$$

$$= \frac{(1+j1)(-2+j1)(3-j1)(1+j2)}{(1e^{j2(90^\circ)})\,(\underbrace{j1}_{1e^{j90^\circ}})\,(24-j8)}$$

$\sqrt{1+4}\; e^{j\tan^{-1}\left(\frac{2}{1}\right)}$

$$= \frac{(\sqrt{2}\,e^{j45^\circ})\,(\sqrt{4+1}\,e^{j(180-\tan^{-1}(\frac{1}{2}))})\,(\sqrt{9+1}\,e^{-j\tan^{-1}(\frac{1}{3})})\,(\quad)}{(1e^{j2(90^\circ)})(1e^{j90^\circ})(\sqrt{24^2+64}\;e^{-j\tan^{-1}(\frac{8}{24})})}$$

$$|H(j1)| = \frac{(\sqrt{2})(\sqrt{4+1})(\sqrt{9+1})(\sqrt{1+4})}{(1)\,(1)\,(\sqrt{24^2+64})}$$

$\angle H(j1) = \angle \text{top} - \angle \text{bottom}$

$= [45^\circ + (180 - \tan^{-1}(\tfrac{1}{2})) - (\tan^{-1}(\tfrac{1}{3})) + \tan^{-1}(\tfrac{2}{1})] - [2(90) + 90^\circ - \tan^{-1}(\tfrac{8}{24})]$

$\boxed{\text{Example 28}}$ (Mastering complex #3) $\boxed{1550}$

$\boxed{\omega = 5}$

$$H(j5) = \frac{(j5+1)\,(j5-2)\,(3-j5)\,(-25+j10+2)}{(5e^{j90^\circ})^2\,(-25+j5+1)\,(-25-j40+25)}$$

$$= \frac{(1+j5)\,(-2+j5)\,(3-j5)\,(-23+j10)}{(25e^{j2(90^\circ)})\,(-24+j5)\,(-j40)}$$

$40e^{-j90^\circ}$

$40 \angle -90^\circ$

$\sqrt{23^2+100}\,e^{j(180^\circ - \mathrm{Tan}^{-1}(\frac{10}{23}))}$

$$= \frac{\left(\sqrt{1+25}\,e^{j\,\mathrm{Tan}^{-1}(\frac{5}{1})}\right)\left(\sqrt{4+25}\,e^{j(180-\mathrm{Tan}^{-1}(\frac{5}{2}))}\right)\left(\sqrt{9+25}\,e^{-j\,\mathrm{Tan}^{-1}(\frac{5}{3})}\right)(\;\;)}{\left(25e^{j2(90^\circ)}\right)\left(\sqrt{24^2+25}\,e^{j(180-\mathrm{Tan}^{-1}(\frac{5}{24}))}\right)\left(40e^{-j90^\circ}\right)}$$

$$|H(j5)| = \frac{(\sqrt{1+25})(\sqrt{4+25})(\sqrt{9+25})(\sqrt{23^2+100})}{(25)(\sqrt{24^2+25})(40)} \leftarrow \frac{|top|}{|bottom|}$$

$$\angle H(j5) = \angle top - \angle bottom$$

$$= \left[\mathrm{Tan}^{-1}(\tfrac{5}{1}) + (180 - \mathrm{Tan}^{-1}(\tfrac{5}{2})\right] + \left(-\mathrm{Tan}^{-1}(\tfrac{5}{3})\right) + \left(180^\circ - \mathrm{Tan}^{-1}(\tfrac{10}{23})\right)\Big]$$

$$- \left[2(90^\circ) + (180 - \mathrm{Tan}^{-1}(\tfrac{5}{24})) + (-90^\circ)\right]$$

$\boxed{\text{Example 28}}$ (Mastering complex #'s) $\boxed{1570}$

$\boxed{\omega = 10}$

$$H(j10) = \frac{(j10+1)(j10-2)(3-j10)(-100+j20+2)}{(10e^{j90°})^2(-100+j10+1)(-100-j80+25)}$$

$10e^{j90°}$ 10 90° 10 2 3 10 20 100

$$\cong \frac{(j10+0)(-2+j10)(3-j10)(-100+j20+0)}{(100e^{j2(90°)})(-100+0+0)(-75-j80)}$$

$180-Tan^{-1}\left(\frac{20}{100}\right)$

$\sqrt{10^4+400}\,e^{j(\;)}$

180° 75 80

$100e^{j180°}$ 100

$$\cong \frac{(10e^{j90})(\sqrt{4+100}\,e^{j(180-Tan^{-1}(\frac{10}{2}))})(\sqrt{9+100}\,e^{-jTan^{-1}(\frac{10}{3})})(\;)}{(100e^{j2(90°)})(100e^{j180°})(\sqrt{75^2+80^2}\,e^{j(-180+Tan^{-1}(\frac{80}{75}))})}$$

$$|H(j10)| = \frac{|top|}{|bottom|} = \frac{(10)(\sqrt{4+100})(\sqrt{9+100})(\sqrt{10^4+400})}{(100)(100)(\sqrt{75^2+80^2})}$$

$$\angle H(j10) = \angle top - \angle bottom$$

$$= \left[90° + (180-Tan^{-1}(\tfrac{10}{2})) + (-Tan^{-1}(\tfrac{10}{3})) + (180-Tan^{-1}(\tfrac{20}{100}))\right]$$

$$-\left[2(90°) + (180°) + (-180+Tan^{-1}(\tfrac{80}{75}))\right]$$

Example 28 (Mastering complex #'s) 1580

$\boxed{\omega = 1000}$

too much work

$$H(j10^3) = \frac{(j10^3+1)(j10^3-2)(3-j10^3)(-10^6+j2\times10^3+2)}{(10^3 e^{j90^\circ})^2(-10^6+j10^3+1)(-10^6-j8\times10^3+25)}$$

$$\cong \frac{(j10^3)(j10^3)(-j10^3)(-10^6)}{(10^6 e^{j2(90^\circ)})(-10^6)(-10^6)}$$

$$\cong \frac{(10^3 e^{j90^\circ})(10^3 e^{j90^\circ})(10^3 e^{-j90^\circ})(10^6 e^{j180^\circ})}{(10^6 e^{j2(90^\circ)})(10^6 e^{j180^\circ})(10^6 e^{j180^\circ})}$$

$$|H(j10^3)| = \frac{|top|}{|bottom|} \cong \frac{(10^3)\cancel{(10^3)}\cancel{(10^3)}\cancel{(10^6)}}{(10^6)\cancel{(10^6)}\cancel{(10^6)}} = 10^{-3}$$

$$\angle H(j10^3) = \angle top - \angle bottom \cong [\cancel{90+90} - 90 + \cancel{180^\circ}] - [\cancel{2(90^\circ)} + 180 + \cancel{180^\circ}] = -270^\circ$$

Note: For s large

look here!

$$H(s) \cong \frac{(s)(s)(-s)(s^2)}{s^2(s^2)(s^2)} = \frac{-s^5}{s^6} = \frac{-1}{s}$$

from which $H(j10^3) \cong \dfrac{-1}{j10^3} = \dfrac{1e^{j180^\circ}}{10^3 e^{j90}}$

$$= 10^{-3} e^{j90^\circ}$$

$$= 10^{-3} e^{-j270^\circ}$$

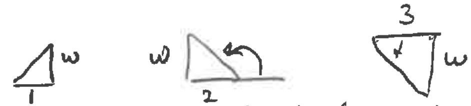
$\Longrightarrow$ $|H(j10^3)| \cong 10^{-3}$

$\angle H(j10^3) \cong 90^\circ$ or -270° !

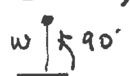
Example 28 (Mastering complex #'s)

$$H(s) = \frac{(s+1)(s-2)(3-s)(s^2+2s+2)}{s^2(s^2+s+1)(s^2-8s+25)}$$

Lets compute $H(j\omega)$



$$H(j\omega) = \frac{(1+j\omega)(-2+j\omega)(3-j\omega)(2-\omega^2+j2\omega)}{(\omega e^{j90^\circ})^2(1-\omega^2+j\omega)(25-\omega^2-j8\omega)}$$



From this, we have

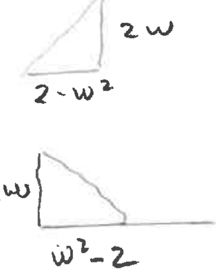
$$|H(j\omega)| = \frac{|top|}{|bottom|} = \frac{(\sqrt{1+\omega^2})(\sqrt{4+\omega^2})(\sqrt{9+\omega^2})(\sqrt{(2-\omega^2)^2+4\omega^2})}{(\omega^2)(\sqrt{(1-\omega^2)^2+\omega^2})(\sqrt{(25-\omega^2)^2+64\omega^2})}$$

$\angle H(j\omega) = \angle top - \angle bottom$

$$\angle top = \angle 1+j\omega + \angle -2+j\omega + \angle 3-j\omega + \angle 2-\omega^2+j2\omega$$

$$= \tan^{-1}\left(\frac{\omega}{1}\right) + \left[180 - \tan^{-1}\left(\frac{\omega}{2}\right)\right] - \tan^{-1}\left(\frac{\omega}{3}\right) + \angle 2-\omega^2+j2\omega$$

$$\angle 2-\omega^2+j2\omega = \begin{cases} \tan^{-1}\left(\dfrac{2\omega}{2-\omega^2}\right) & 2>\omega^2 \; (\sqrt{2}>\omega) \\ 180^\circ - \tan^{-1}\left(\dfrac{2\omega}{\omega^2-2}\right) & 2<\omega^2 \; (\sqrt{2}<\omega) \end{cases}$$



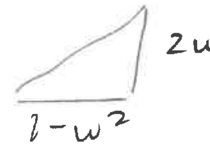
$$\angle bottom = 2(90^\circ) + \angle 1-\omega^2+j\omega + \angle 25-\omega^2-j8\omega$$

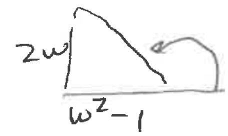
lets now address these 2 terms - - -

$\boxed{\text{Example 28}}$ (Mastering Complex #'s) $\boxed{1600}$

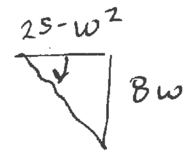
Mon 3/18/24

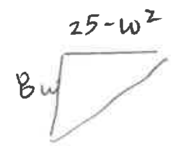
$$\angle\, 1-\omega^2+j2\omega = \begin{cases} \tan^{-1}\left(\dfrac{2\omega}{1-\omega^2}\right) & (\omega \le 1) \\ 180^\circ - \tan^{-1}\left(\dfrac{2\omega}{\omega^2-1}\right) & (\omega \ge 1) \end{cases}$$





$$\angle\, 25-\omega^2-j8\omega = \begin{cases} -\tan^{-1}\left(\dfrac{8\omega}{25-\omega^2}\right) & (\omega \le 5) \\ -180^\circ + \tan^{-1}\left(\dfrac{8\omega}{25-\omega^2}\right) & (\omega \ge 5) \end{cases}$$





$\boxed{\text{Example 28}}$ (Mastering complex #'s) $\boxed{1610}$

Mon 3-18-24

How is it that $z = x + jy$

can be written in polar form $z = |z|\, e^{j\angle z}$?

This great fact we owe to the great Euler (1748).

$$z = x + jy = \text{Re}\, z + j\, \text{Im}\, z$$

$$= |z| \cos\angle z + j\, |z| \sin\angle z$$

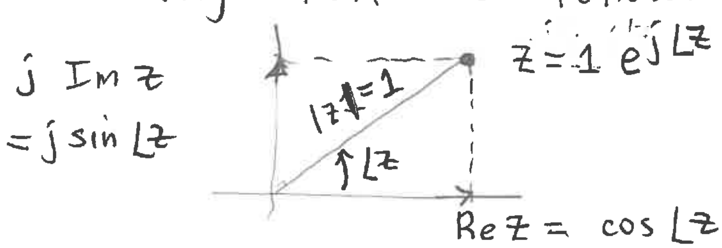
$$= |z| \left[\cos\angle z + j \sin\angle z \right]$$

$$= |z|\, e^{j\angle z}$$

It was Euler that 1st showed (1748, via series) that

$$\boxed{e^{j\angle z} = \cos\angle z + j \sin\angle z}$$ ← Euler's Identity

While he used series to prove this, it follows intuitively from the following picture:



$z = 1\, e^{j\angle z} = \cos\angle z + j \sin\angle z$

From Euler's identity we also have

$$e^{-j\angle z} = \cos(-\angle z) + j\sin(-\angle z) = \cos\angle z - j\sin\angle z$$

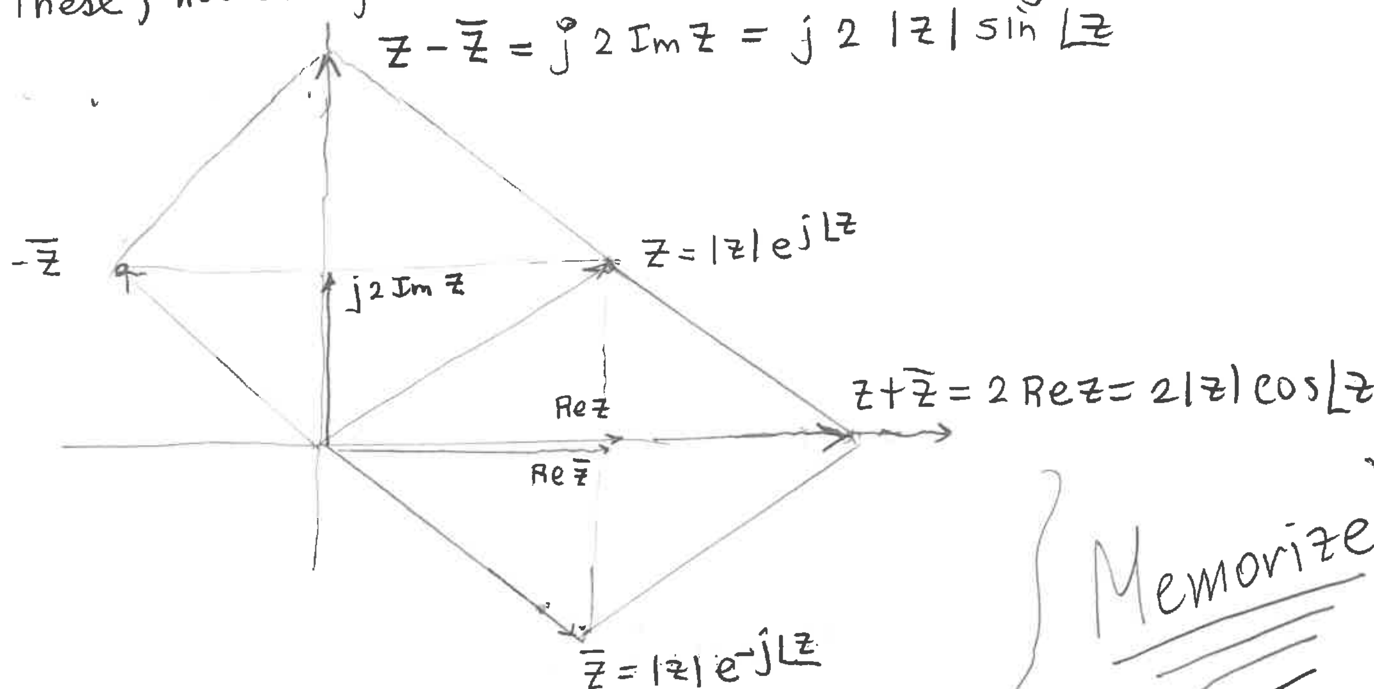
since $\cos(-x) = \cos x$ & $\sin(-x) = -\sin x$

From the above we then get

$$e^{j\angle z} + e^{-j\angle z} = 2\cos\angle z = 2\,\mathrm{Re}\, e^{j\angle z}$$

$$e^{j\angle z} - e^{-j\angle z} = j2\sin\angle z = j2\,\mathrm{Im}\, e^{j\angle z}$$

These, however, follow from the following compelling picture =



Memorize!

$$\boxed{\begin{aligned} z+\bar{z} &= 2\,\mathrm{Re}\,z = 2|z|\cos\angle z \\ z-\bar{z} &= j2\,\mathrm{Im}\,z = j2|z|\sin\angle z \end{aligned}}$$

← Euler Identities

Example 28 (Mastering complex #'s) 1630

To see how the above Euler identities can be applied to sinusoids, let

$$\boxed{\begin{aligned} z &= A\, e^{(\sigma + j\omega)t} \\ &= |A|\, e^{j\angle A}\, e^{\sigma t}\, e^{j\omega t} \\ &= \underbrace{|A|\, e^{\sigma t}}_{|z|}\, e^{j\overbrace{(\omega t + \angle A)}^{\angle z}} \end{aligned}}$$

From this it follows that

$$\boxed{\begin{aligned} |A|\, e^{\sigma t} \cos(\omega t + \angle A) &= |z| \cos \angle z \\ &= \operatorname{Re} z \\ &= \frac{z + \bar{z}}{2} \end{aligned}}$$

&

$$\boxed{\begin{aligned} |A|\, e^{\sigma t} \sin(\omega t + \angle A) &= |z| \sin \angle z \\ &= \operatorname{Im} z \\ &= \frac{z - \bar{z}}{j2} \end{aligned}}$$

Example 28

(Mastering Complex #'s) 1640

Now lets show how the above can be used to add sinusoids!

Consider

$$f(t) = A_1 \cos(wt + \theta_1) + A_2 \cos(wt + \theta_2) + A_3 \cos(wt + \theta_3)$$

We want to add these to get

$$f(t) = A \cos(wt + \theta)$$

lets find A & θ. (using complex #'s --- NO TRIG!)

$$f(t) = \mathrm{Re}\left[A_1 e^{j(wt+\theta_1)} + A_2 e^{j(wt+\theta_2)} + A_3 e^{j(wt+\theta_3)} \right]$$

$$= \mathrm{Re}\left\{ \left(A_1 e^{j\theta_1} + A_2 e^{j\theta_2} + A_3 e^{j\theta_3} \right) e^{jwt} \right\}$$

let this be $A e^{j\theta}$

$$= \mathrm{Re}\left\{ A e^{j(wt+\theta)} \right\}$$

$$= A \cos(wt + \theta)$$

To find A & θ we just need to solve

$$A e^{j\theta} = A_1 e^{j\theta_1} + A_2 e^{j\theta_2} + A_3 e^{j\theta_3}$$

i.e. add this, find magnitude & angle

A = magnitude θ = angle!

Problem 28

1650

1 Find A & θ in $f(t) = A\cos(wt+\theta)$

where

$$f(t) = 1\cos(wt + 30^\circ) + 2\cos(wt + 45^\circ) + 3\cos(wt + 60^\circ)$$

Hint:
see pg 1640

add:
$z = 1e^{j30^\circ} + 2e^{j45^\circ} + 3e^{j60^\circ}$
Find $|z|$ & $\angle z$

2 Consider

$$H(s) = \frac{s^3\ (s+1)\ (10-s)\ (s^2+8s+25)}{(s+3)\ (s^2-s-1)\ (s^2-6s+18)}$$

a) Compute $|H(jw)|$, $\angle H(jw)$

Hint: Draw pictures
See pages 1590 - 1600

b) Estimate $|H(jw)|$ & $\angle H(jw)$ for $w = 0.1$

c) " " " " $w = 1$

d) " " " " $w = 100$

3 Consider
$$z = \frac{(-1+j1)^2\ (-\sqrt{3}-j1)\ (4-j3)^3\ (6e^{j90^\circ})\ (8)}{(j2)^3\ (2+j2\sqrt{3})\ (-6+j8)^4\ (-4)\ (-2-j3)}$$

Determine $|z|$, $\angle z$, $\text{Re}\, z$, $\text{Im}\, z$.

Example 29

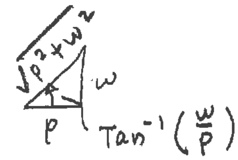
Transfer Function (Frequency Responses)

For each of the following transfer functions compute & plot the magnitude response $|H(j\omega)|$ and the phase response $\angle H(j\omega)$.

Note: This is useful because of the so-called Method of the Transfer Function (MOTF)!

1 $$\boxed{H(s) = \frac{b}{s+p}} \qquad b, p > 0$$

$$H(j\omega) = \frac{b}{j\omega + p}$$



$$|H(j\omega)| = \frac{|\text{top}|}{|\text{bottom}|} = \frac{b}{\sqrt{\omega^2+p^2}}$$

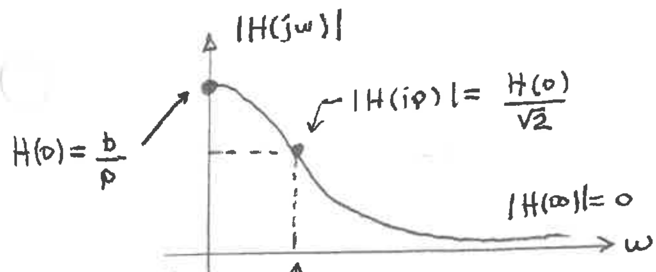
Note:

$H(0) = \frac{b}{p}$

$H(jp) = \frac{b}{jp+p} = \frac{b/p}{1+j1} = \frac{H(0)}{\sqrt{2}\, e^{j45^\circ}} = \frac{H(0)}{\sqrt{2}} e^{-j45^\circ}$

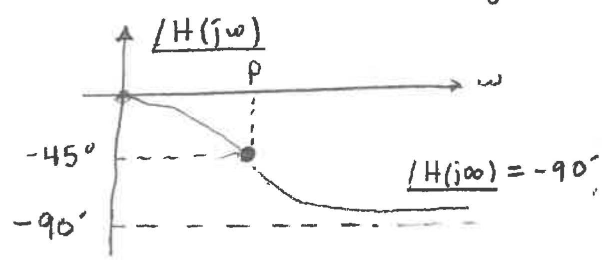
$H(j\omega) \underset{\omega \text{ large}}{\cong} \frac{b}{j\omega} = \frac{b}{\omega e^{j90^\circ}} = \frac{b}{\omega} e^{-j90^\circ}$

$H(\infty) = 0$



$\omega_{3dB} = p$ since $20\log_{10}|H(jp)| = 20\log_{10}\left(\frac{H(0)}{\sqrt{2}}\right) = 20\log_{10} H(0) - 20\log_{10}\sqrt{2}$
$= 20\log_{10} H(0) - 3$

$$\begin{aligned}\angle H(j\omega) &= \angle\text{top} - \angle\text{bottom} \\ &= 0 - \text{Tan}^{-1}\left(\frac{\omega}{p}\right) \\ &= -\text{Tan}^{-1}\left(\frac{\omega}{p}\right)\end{aligned}$$

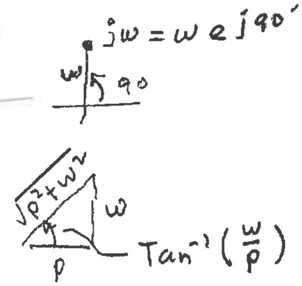


Because of this magnitude response, H is called a low pass filter (LPF).

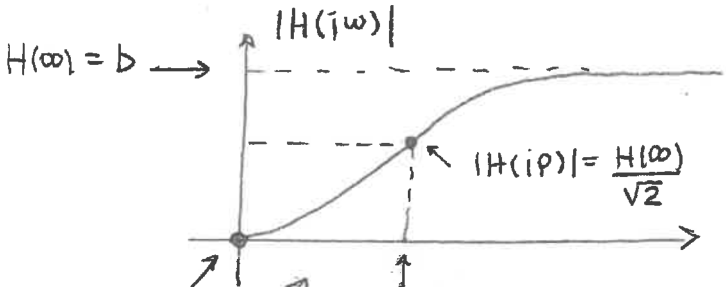
Example 29

$\boxed{2}$ $\boxed{H(s) = \dfrac{b\,s}{s+p}}$ $\quad b, p > 0$

$$H(j\omega) = \frac{b\,j\omega}{j\omega + p}$$



$$|H(j\omega)| = \frac{|top|}{|bottom|} = \frac{b\,\omega}{\sqrt{\omega^2 + p^2}}$$



$\omega_{3dB} = p$ since $20\log_{10}|H(jp)|$

$= 20\log_{10} H(\infty) - 20\log_{10}\sqrt{2}$

$= 20\log_{10} H(\infty) - 3$

Note:

$H(0) = 0$

$H(\infty) = b$

$H(jp) = \dfrac{bjp}{jp+p}$

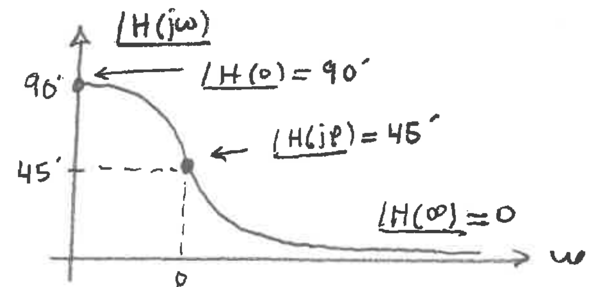
$= \dfrac{jb}{1+j1}$

$= \dfrac{be^{j90°}}{\sqrt{2}\,e^{j45°}}$

$= \dfrac{H(\infty)}{\sqrt{2}}\, e^{j(90-45)°}$ $\quad 45°$

$$\underline{/H(j\omega)} = \underline{/top} - \underline{/bottom}$$

$$= 90° - Tan^{-1}\left(\frac{\omega}{p}\right)$$

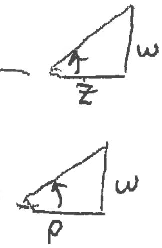


Because of this magnitude response, H is called a high pass filter (HPF).

Example 29

$\boxed{3}$ $$\boxed{H(s) = g\left[\frac{s+z}{s+p}\right]} \qquad g, z, p > 0$$

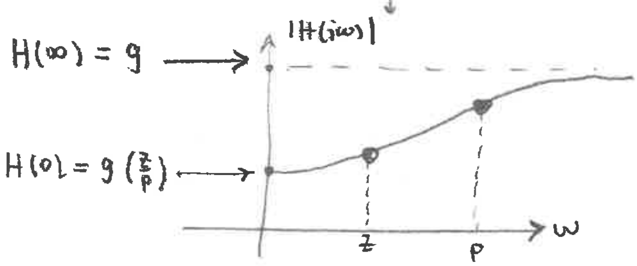
$$H(j\omega) = g\left[\frac{j\omega+z}{j\omega+p}\right]$$



$$|H(j\omega)| = \frac{|\text{top}|}{|\text{bottom}|} = g\,\frac{\sqrt{\omega^2+z^2}}{\sqrt{\omega^2+p^2}}$$

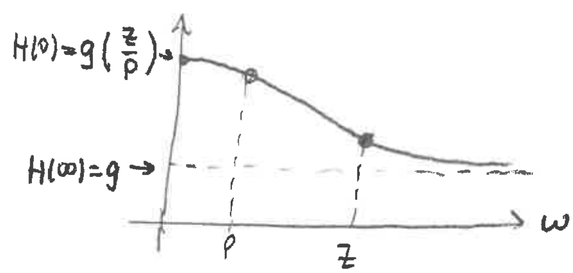
Note: $H(0) = g\left(\frac{z}{p}\right)$ $\qquad H(\infty) = g$

Suppose $\boxed{z < p}$



H is a high pass filter (HPF) when $z < p$

Suppose $\boxed{z > p}$



H is a low pass filter (LPF) when $z > p$

$$\angle H(j\omega) = \angle\text{top} - \angle\text{bottom} = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right)$$

Lets find when $\angle H(j\omega)$ is max or min!

Recall: $$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Example 29

$$\frac{d\angle H(j\omega)}{d\omega} = \left(\frac{1}{1+\left(\frac{\omega}{z}\right)^2}\right)\left(\frac{1}{z}\right) - \left(\frac{1}{1+\left(\frac{\omega}{p}\right)^2}\right)\frac{1}{p} = 0$$

↓ multiply top & bottom by z^2 ↓ multiply top & bottom by p^2

$$\Longrightarrow \quad \frac{z}{z^2+\omega^2} = \frac{p}{p^2+\omega^2}$$

$$\Longrightarrow \quad z(p^2+\omega^2) = p(z^2+\omega^2)$$

$$\Longrightarrow \quad zp^2 + z\omega^2 = pz^2 + p\omega^2$$

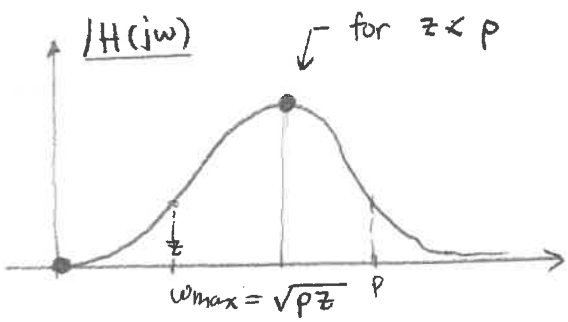
$$\Longrightarrow \quad (p-z)\omega^2 = zp^2 - pz^2 = zp(p-z)$$

$$\Longrightarrow \quad \omega^2 = pz$$

$$\boxed{\omega_{@\text{xtrema}} = \sqrt{pz}}$$

$$\angle H(j\sqrt{pz}) = \tan^{-1}\left(\frac{\sqrt{pz}}{z}\right) - \tan^{-1}\left(\frac{\sqrt{pz}}{p}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{p}{z}}\right) - \tan^{-1}\left(\sqrt{\frac{z}{p}}\right)$$



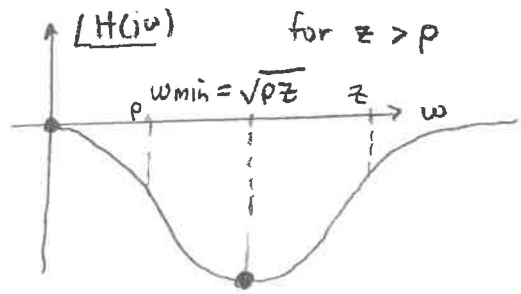
When $z < p$, $\angle H(j\omega) \geq 0$

& we say that H

is a Lead filter

(as well as a high pass filter (HPF))

because of magnitude shape



When $z > p$, $\angle H(j\omega) \leq 0$

& we say that H

is a Lag Filter.

(as well as a low pass filter (LPF))

because of magnitude shape

Example 29

4 $$\boxed{H(s) = g\left[\frac{s}{s+p_1}\right]\left[\frac{p_2}{s+p_2}\right]}$$

$g, p_1, p_2 > 0$

$(p_1 << p_2)$

↑

(suppose

$10\,p_1 << 0.1\,p_2$

to facilitate plotting of magnitude)

$j\omega = \omega e^{j90°}$

$$H(j\omega) = g\left[\frac{j\omega}{j\omega+p_1}\right]\left[\frac{p_2}{j\omega+p_2}\right]$$

$$|H(j\omega)| = \frac{|\text{top}|}{|\text{bottom}|} = g\left[\frac{\omega}{\sqrt{\omega^2+p_1^2}}\right]\left[\frac{p_2}{\sqrt{\omega^2+p_2^2}}\right]$$

Note: $H(0) = 0 \qquad H(\infty) = 0$

$$|H(jp_1)| = \left|g\left[\frac{jp_1}{jp_1+p_1}\right]\left[\frac{p_2}{jp_1+p_2}\right]\right|$$

$$= \left|g\left[\frac{j1}{1+j1}\right]\left[\frac{p_2}{p_2+jp_1}\right]\right|$$

$p_1 << p_2$

$$\cong g\,\frac{1}{\sqrt{2}}\left[\frac{p_2}{p_2+0}\right]$$

$$\cong \frac{g}{\sqrt{2}}$$

$$|H(j10p_1)| = \left|g\left[\frac{j10p_1}{j10p_1+p_1}\right]\left[\frac{p_2}{j10p_1+p_2}\right]\right|$$

$$\cong g\left|\frac{j10p_1}{j10p_1+0}\right|\left|\frac{p_2}{0+p_2}\right|$$

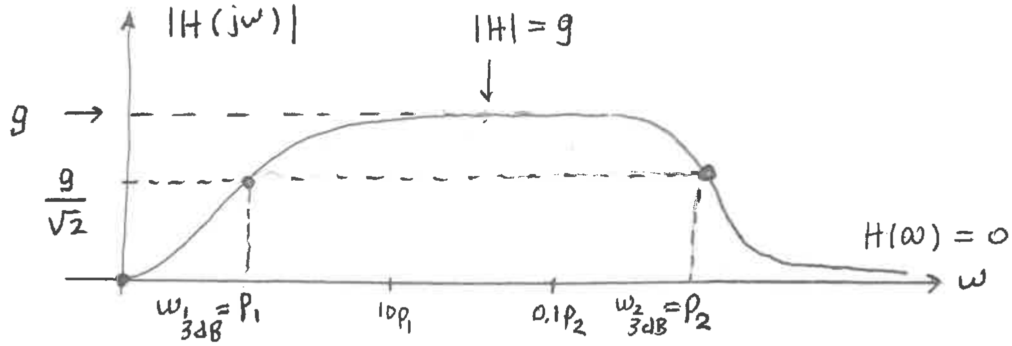
$$\cong g$$

$$|H(jp_2)| = \left|g\left[\frac{jp_2}{jp_2+p_1}\right]\left[\frac{p_2}{jp_2+p_2}\right]\right|$$

$$\cong g\left|\frac{jp_2}{jp_2+0}\right|\left|\frac{1}{1+j1}\right|$$

$$\cong \frac{g}{\sqrt{2}}$$

Example 29



Here, H is called a <u>band pass filter (BPF)</u>.

It is used to pass (or amplify) frequencies between the 3dB frequencies p_1 & p_2 !

$\underline{/H(j\omega)} = \underline{\angle top} - \underline{/bottom}$

$$= 90^\circ - \left[\operatorname{Tan}^{-1}\left(\frac{\omega}{p_1}\right) + \operatorname{Tan}^{-1}\left(\frac{\omega}{p_2}\right)\right]$$

<u>Note:</u>

$\underline{/H(0)} = 90 - 0 - 0 = 90^\circ$

$\underline{/H(jp_1)} = 90 - \underbrace{\operatorname{Tan}^{-1}(1)}_{45} - \underbrace{\operatorname{Tan}^{-1}\left(\frac{p_1}{p_2}\right)}_{\sim 0^\circ \text{ since } p_1 << p_2} \cong 45^\circ$

$\underline{/H(j10p_1)} = 90 - \underbrace{\operatorname{Tan}^{-1}(10)}_{\sim 90^\circ} - \underbrace{\operatorname{Tan}^{-1}\left(\frac{10p_1}{p_2}\right)}_{\sim 0}$

$\cong 0^\circ$

$\underline{/H(jp_2)} = 90 - \underbrace{\operatorname{Tan}^{-1}\left(\frac{p_2}{p_1}\right)}_{\sim 90^\circ \text{ since } p_2 >> p_1} - \underbrace{\operatorname{Tan}^{-1}(1)}_{45^\circ}$

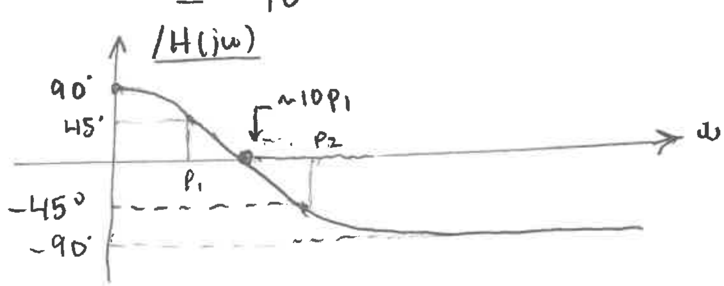
$\cong 90 - 90^\circ - 45^\circ$

$\cong -45^\circ$

$\underline{/H(j10p_2)} = 90 - \underbrace{\operatorname{Tan}^{-1}\left(\frac{10p_2}{p_1}\right)}_{\sim 90^\circ} - \underbrace{\operatorname{Tan}^{-1}(10)}_{\sim 90^\circ}$

$\cong 90 - 90 - 90^\circ$

$\cong -90^\circ$



Example 29 — 1720

[5] $$\boxed{H(s) = \frac{g}{p}\, s \left[\frac{p}{s+p}\right]^2}$$ $g, p > 0$

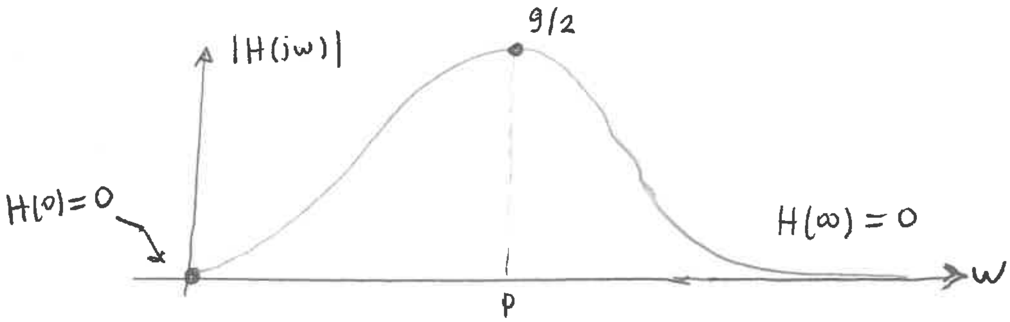
$jw = w e^{j90}$

$$H(jw) = \frac{g}{p}\, jw \left[\frac{p}{jw+p}\right]^2$$

$$|H(jw)| = \frac{|top|}{|bottom|} = \frac{\frac{g}{p} w \; p^2}{\left(\sqrt{w^2+p^2}\right)^2} = \frac{g p w}{w^2+p^2}$$

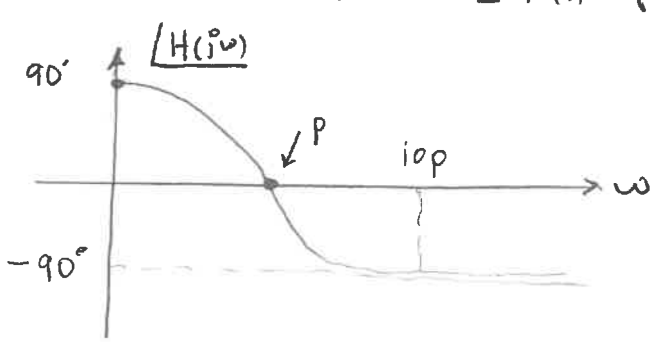
Note: $H(0) = 0$ $\quad H(\infty) = 0$

$$|H(jp)| = \frac{g\, p^2}{p^2+p^2} = \frac{g}{2}$$



This H may also be called a band pass filter (BPF).

$$\angle H(jw) = \angle top - \angle bottom$$
$$= 90^\circ - 2\,\mathrm{Tan}^{-1}\left(\frac{w}{p}\right)$$



Note:

$\angle H(0) = 90^\circ$

$\angle H(jp) = 90 - 2\,\mathrm{Tan}^{-1}(1)$
$= 90 - 2(45^\circ)$
$= 0^\circ$

$\angle H(j10p) = 90 - 2\,\mathrm{Tan}^{-1}(10)$
$\cong 90 - 2(90^\circ)$
$= -90^\circ$

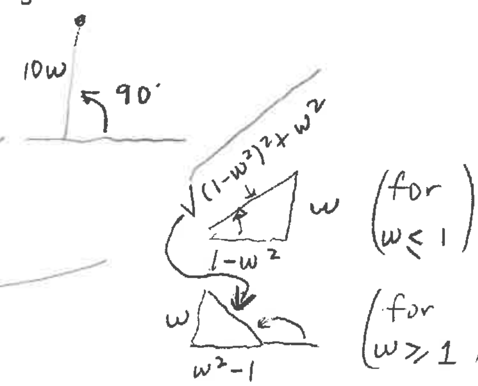
Example 29

6

$$\boxed{H(s) = \frac{10s}{s^2+s+1}}$$

Note: poles $= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$j10w = 10w\,e^{j90^\circ}$



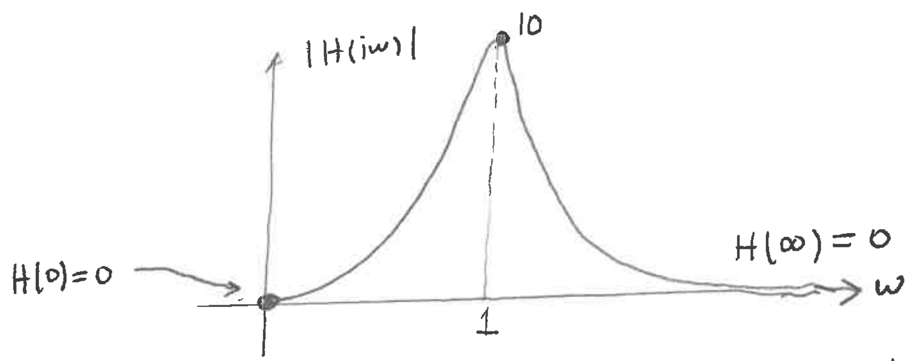
$$H(jw) = \frac{10jw}{-w^2+jw+1} = \frac{j10w}{1-w^2+jw}$$

$$|H(jw)| = \frac{|top|}{|bottom|} = \frac{10w}{\sqrt{(1-w^2)^2+w^2}}$$

Note: $H(0) = 0$

$$H(j1) = \frac{10j1}{-1+j1+1} = 10$$

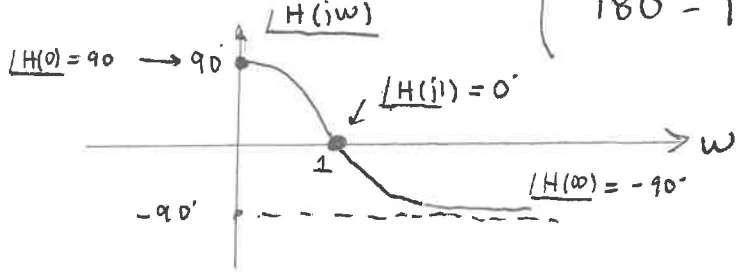
$H(\infty) = 0\,e^{-j90^\circ}$ (since $H \approx \frac{10}{s}$ for large s)



This H is called a resonant band pass filter (BPF).

$$\angle H(jw) = \angle top - \angle bottom$$

$$= 90^\circ - \begin{cases} \text{Tan}^{-1}\left(\frac{w}{1-w^2}\right) & \text{if } w \in [0,1] \\ 180^\circ - \text{Tan}^{-1}\left(\frac{w}{w^2-1}\right) & \text{if } w \in [1,\infty) \end{cases}$$



Note: cause $H \approx 10s$ for small s!

$\angle H(0) = 90$

$\angle H(j1) = 90 - 90^\circ = 0^\circ$

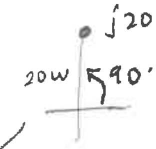
$\angle H(\infty) = 90 - (180 - 0)$

$= -90^\circ$ ← cause $H \approx \frac{10}{s}$ for large s

Example 29

[7] $$\boxed{H(s) = \frac{20s}{s^2 + 10s + 125}}$$

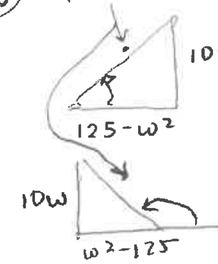
poles $= -5 \pm j10$



$j20w = 20w e^{j90}$

$$H(jw) = \frac{20jw}{-w^2 + 10jw + 125} = \frac{j20w}{(125 - w^2 + j10w)}$$

mag(bottom) $= \sqrt{(125-w^2)^2 + 100w^2}$



for $w \in [0, \sqrt{125}]$

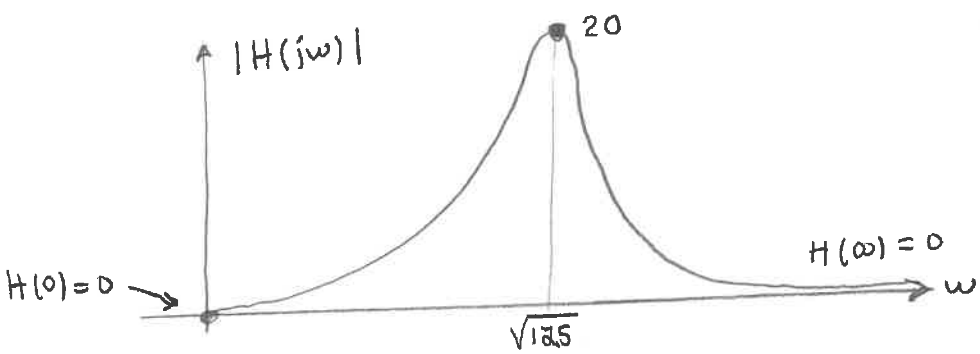
for $w \in [\sqrt{125}, \infty)$

$$|H(jw)| = \frac{|top|}{|bottom|} = \frac{20w}{\sqrt{(w^2-125)^2 + 100w^2}}$$

<u>Note:</u> $H(0) = 0$ $\qquad H(s) \overset{s\,small}{\cong} \frac{20s}{125}$

$$H(j\sqrt{125}) = \frac{200j\sqrt{125}}{-125 + 10j\sqrt{125} + 125} = \frac{200}{10} = 20$$

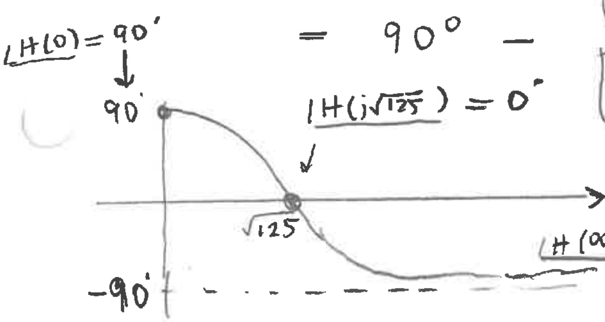
$H(\infty) = 0\, e^{-j90}$ (since $H \approx \frac{20}{s}$ for large s)



This H is called a <u>resonant bandpass filter (BPF)</u>.

$$\angle H(jw) = \angle top - \angle bottom$$

$$= 90^\circ - \begin{cases} \tan^{-1}\left(\frac{10w}{125-w^2}\right) & \text{if } w \in [0, \sqrt{125}] \\ 180^\circ - \tan^{-1}\left(\frac{10w}{w^2-125}\right) & \text{if } w \in [\sqrt{125}, \infty) \end{cases}$$



<u>Note:</u> $\angle H(0) = 90^\circ$ since $H \approx 20s$ for small s!

$\angle H(j\sqrt{125}) = 90 - 90^\circ = 0^\circ$

$\angle H(\infty) = 90 - (180 - 0^\circ) = -90^\circ$ since $H \cong \frac{20}{s}$ for large s

$\boxed{\text{Example 29}}$ $\boxed{1750}$

undamped natural frequency → ω_n, damping factor → ζ

$g, \omega_n, \zeta \geq 0$

$\boxed{8}$

$$\boxed{H(s) = g\left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]}$$

$$\text{poles} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad (\text{for } \zeta \geq 1)$$

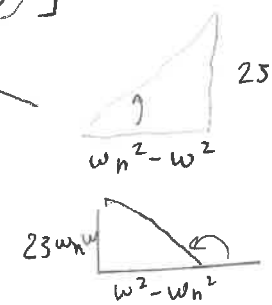
$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad (\text{for } \zeta \in [0, 1])$$

$$(= \pm j\omega_n \text{ when } \zeta = 0)$$

This is why ω_n is called the undamped natural frequency!

$$H(j\omega) = g\left[\frac{\omega_n^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}\right]$$

$$= g\left[\frac{\omega_n^2}{\omega_n^2 - \omega^2 + j\,2\zeta\omega_n\omega}\right]$$



if $\omega \in [0, \omega_n]$

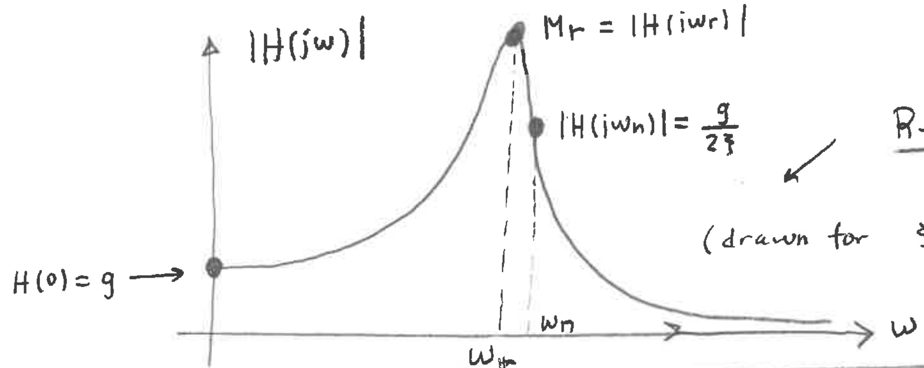
if $\omega \in [\omega_n, \infty)$

$$|H(j\omega)| = \frac{|\text{top}|}{|\text{bottom}|}$$

$$= g\left[\frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}\right]$$

<u>Note:</u> $H(0) = g$ $\qquad H(j\omega_n) = \dfrac{g}{j2\zeta} = \dfrac{g}{2\zeta}e^{-j90^\circ}$ $\qquad H(\infty) = 0\,e^{-j180^\circ}$

since $H \approx \dfrac{g\omega_n^2}{s^2}$ for large s



Resonant Band Pass Filter (BPF)

(drawn for $\zeta \in [0, \frac{1}{\sqrt{2}})$)

From calculus (or clever algebra)

$$\boxed{\omega_r = \begin{cases} 0 & \zeta \in [\frac{1}{\sqrt{2}}, \infty) \\ \omega_n\sqrt{1-2\zeta^2} & \zeta \in [0, \frac{1}{\sqrt{2}}] \end{cases}}$$

$$\boxed{M_r = |H(j\omega_r)| = \begin{cases} H(0) = g & \zeta \in [\frac{1}{\sqrt{2}}, \infty) \\ \dfrac{g}{2\zeta\sqrt{1-\zeta^2}} & \zeta \in [0, \frac{1}{\sqrt{2}}] \end{cases}}$$

Example 29

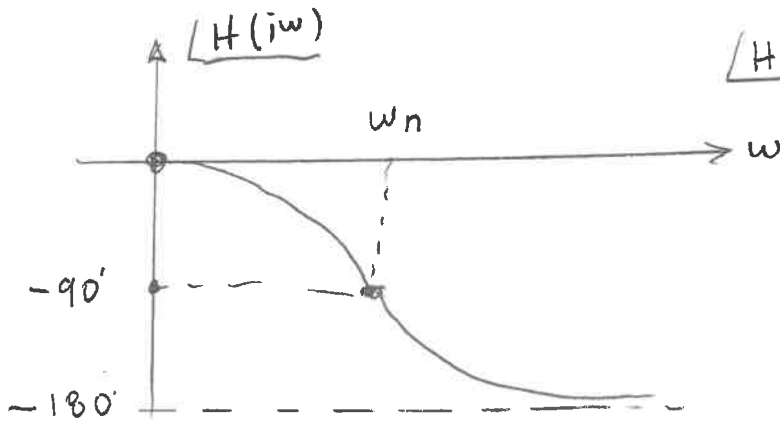
$$\angle H(j\omega) = \angle \text{top} - \angle \text{bottom}$$

$$= 0^\circ - \begin{cases} \tan^{-1}\left(\dfrac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right) & \text{if } \omega \in [0,\omega_n] \\ 180^\circ - \tan^{-1}\left(\dfrac{2\zeta\omega_n\omega}{\omega^2-\omega_n^2}\right) & \text{if } \omega \in [\omega_n,\infty) \end{cases}$$

Note: $\angle H(0) = 0$ since $H(0) = g > 0$

$\angle H(j\omega_n) = -90^\circ$ since $H(j\omega_n) = \frac{g}{2\zeta} e^{-j90^\circ}$

$\angle H(\infty) = -[180 - 0^\circ] = -180^\circ$



Note: One can just plot points using your calculator!

Lets apply the above

$$H(s) = 10\left[\frac{1}{s^2+s+1}\right]$$

$\omega_n^2 = 1 \Rightarrow \omega_n = 1$

$2\zeta\omega_n = 2\zeta(1) = 1 \Rightarrow \zeta = 0.5$

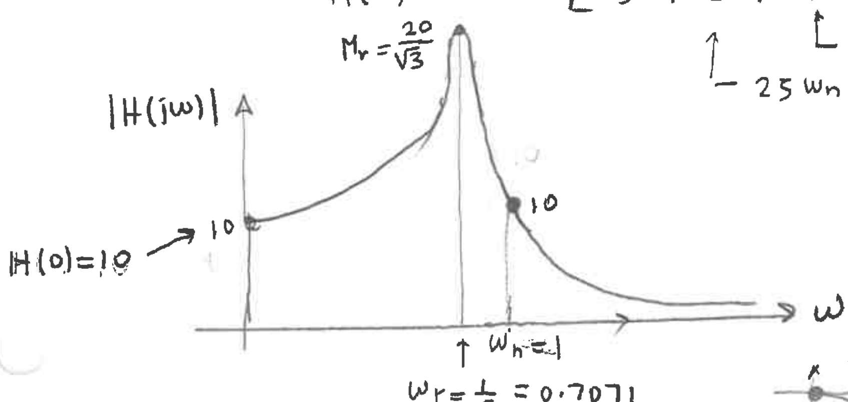
- Need not use the formulae.

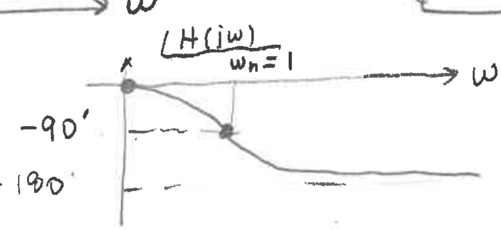
$$\Rightarrow \boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

$$= (1)\sqrt{1-2(\tfrac{1}{4})} = \frac{1}{\sqrt{2}}$$

$$\boxed{M_r = \frac{g}{2\zeta\sqrt{1-\zeta^2}}} = \frac{10}{2(\frac{1}{2})\sqrt{1-\frac{1}{4}}}$$

$$= \frac{20}{\sqrt{3}}$$





Problem 29 (Transfer Function Frequency Response) 1770

For each of the following transfer functions compute & plot the magnitude response $|H(j\omega)|$ & phase response $\angle H(j\omega)$.

1 $H(s) = \dfrac{10}{s+2}$

2 $H(s) = \dfrac{60s}{s+3}$

3 (a) $H(s) = 4\left[\dfrac{s+1}{s+100}\right]$ (b) $H(s) = 4\left[\dfrac{s+100}{s+1}\right]$

4 $H(s) = 75\left[\dfrac{s}{s+10}\right]\left[\dfrac{10^3}{s+10^3}\right]$

5 $H(s) = 30s\left[\dfrac{10}{s+10}\right]^2$

6 $H(s) = \dfrac{30s}{s^2+2s+101}$ poles = ?

7 $H(s) = \dfrac{36s}{s^2+4s+2504}$ poles = ?

8 $H(s) = 25\left[\dfrac{1}{s^2+0.02s+1}\right]$ poles = ?

9 $H(s) = 42\left[\dfrac{10^6}{s^2+2s+10^6}\right]$ poles = ?

Example 30

Laplace Transform Pairs

(More than you will need in circuits!!!)

1780

$$f(t) \leftrightarrow \mathcal{L}\{f(t)\} = F(s) \triangleq \int_{0^-}^{\infty} f(\tau) e^{-s\tau}$$

unilateral Laplace Transform

Marginally stable: M (step) $\leftrightarrow \dfrac{M}{s}$

drawn for $a>0$

$$\boxed{Me^{-at} \leftrightarrow \frac{M}{s+a}}$$

$$\boxed{e^{-at} f(t) \leftrightarrow F(s+a)}$$

Holds for any a!!!

- $a>0$ — stable
- $a=0$ — marginally stable
- $a<0$ — unstable ($-a>0$ ($a<0$))
- $a=\pm j\omega_0$
- $a=\sigma_0 \pm j\omega_0$

a can be anything in complex plane

---

unstable: $t \leftrightarrow \dfrac{1}{s^2}$

$$\boxed{t f(t) \leftrightarrow -\frac{dF(s)}{ds}}$$

unstable: $\dfrac{t^n}{n!} \leftrightarrow \dfrac{1}{s^{n+1}}$

---

stable for $a>0$:

$$Mte^{-at} \leftrightarrow \frac{M}{(s+a)^2}$$

$$M\frac{t^n}{n!}e^{-at} \leftrightarrow \frac{M}{(s+a)^{n+1}}$$

Marginally stable:

$$e^{j\omega_0 t} \leftrightarrow \frac{1}{s-j\omega_0} \qquad (j\omega_0)$$

$$e^{-j\omega_0 t} \leftrightarrow \frac{1}{s+j\omega_0} \qquad (-j\omega_0)$$

$$\cos\omega_0 t \leftrightarrow \frac{s}{s^2+\omega_0^2} \qquad (j\omega_0,\ -j\omega_0)$$

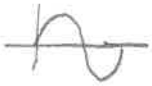
Recall: $\boxed{\dfrac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} = Re\left[e^{j\omega_0 t}\right] = \cos\omega_0 t}$

$$\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} \leftrightarrow \frac{1}{2}\left[\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right] = \frac{1}{2}\left[\frac{s+\cancel{j\omega_0}+s-\cancel{j\omega_0}}{s^2+\omega_0^2}\right] = \frac{s}{s^2+\omega_0^2}$$

Example 30

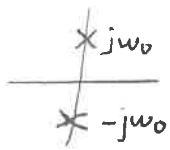
Laplace Transform Pairs

Recall:
$$\boxed{\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{j2} = \operatorname{Im}[e^{j\omega_0 t}] = \sin \omega_0 t}$$



Marginally stable

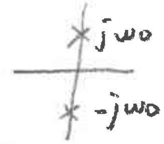
$$\sin \omega_0 t \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2}$$



$$\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{j2} \longleftrightarrow \frac{1}{j2}\left[\frac{1}{s-j\omega_0}-\frac{1}{s+j\omega_0}\right] = \frac{1}{j2}\left[\frac{\cancel{s}+j\omega_0-\cancel{s}+j\omega_0}{s^2+\omega_0^2}\right] = \frac{\omega_0}{s^2+\omega_0^2}$$

Recall: $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$|A|\cos(\omega_0 t + \angle A) \longleftrightarrow |A|\cos\angle A\left[\frac{s-\omega_0\tan\angle A}{s^2+\omega_0^2}\right]$$

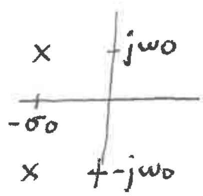


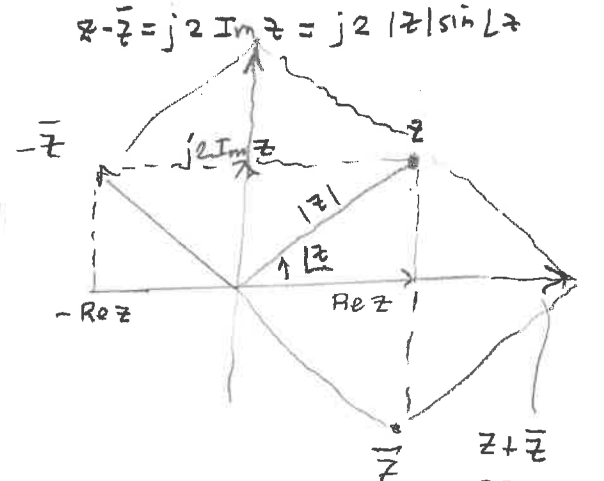
$$= |A|\cos\omega_0 t\cos\angle A - |A|\sin\omega_0 t\sin\angle A \longleftrightarrow |A|\left[\frac{s}{s^2+\omega_0^2}\cos\angle A - \frac{\omega_0\sin\angle A}{s^2+\omega_0^2}\right]$$

(*) $$\boxed{|A|e^{-\sigma_0 t}\cos(\omega_0 t+\angle A) \longleftrightarrow |A|\cos\angle A\left[\frac{s+\sigma_0-\omega_0\tan\angle A}{(s+\sigma_0)^2+\omega_0^2}\right]}$$

(*) $$\boxed{2|A|e^{-\sigma_0 t}\cos(\omega_0 t+\angle A) \longleftrightarrow \frac{A}{s+\sigma_0-j\omega_0} + (*)}$$ $\quad (*) = \dfrac{\bar{A}}{s+\sigma_0+j\omega_0}$

Very useful – comes up when doing partial fraction expansions!





Recall:
$$\boxed{\begin{aligned} z+\bar{z} &= 2\operatorname{Re}z = 2|z|\cos\angle z \\ z-\bar{z} &= j2\operatorname{Im}z = j2|z|\sin\angle z \end{aligned}}$$

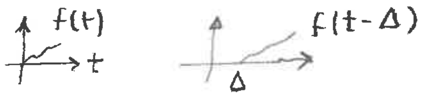
Euler's formulae

$$\boxed{|z|e^{j\angle z} = |z|\left[\cos\angle z + j\sin\angle z\right]}$$

Example 30

Laplace Transform Pairs

1800



time delay system

$$f(t-\Delta) \longleftrightarrow e^{-s\Delta} F(s)$$

$$y(t) = (h * u)(t) = \int_0^t h(\tau)\, u(t-\tau)\, d\tau \longleftrightarrow \boxed{Y(s) = H(s)\, U(s)}$$

convolution of h & u

We will not use this in our circuit/system analysis

Here $u \longleftrightarrow U$

$h \longleftrightarrow H$

we use this to find the output y of a system H with input u (& zero initial conditions!)

- this will involve expanding Y as a partial fraction expansion & then taking inverse transforms term by term

Useful Interpretation:

input to H

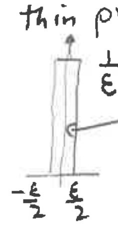
transfer function of a system

output of H

$$\delta(t) \longleftrightarrow 1$$

Dirac Delta function .... (unit impulse function)

mathematical model for a "tall thin pulse" with unit area:



$$\text{Area} = \left(\tfrac{1}{\varepsilon}\right)(\varepsilon) = 1$$

Properties of δ:

$$\int_{0^-}^{0^+} \delta(\tau)\, d\tau = 1 \quad \text{(unit area)}$$

$$\int_{a^-}^{a^+} f(\tau)\, \delta(\tau - a)\, d\tau = f(a)$$

sifting property of δ

we use this to find output y of a system when we have non-zero initial conditions

$$\begin{aligned}
\dot{y} &\longleftrightarrow s\, Y(s) - y_0 \\
\ddot{y} &\longleftrightarrow s^2\, Y(s) - s\, y_0 - \dot{y}_0 \\
\dddot{y} &\longleftrightarrow s^3\, Y(s) - s^2 y_0 - s\, \dot{y}_0 - \ddot{y}_0
\end{aligned}$$

$y_0 \triangleq y(0)$

$\dot{y}_0 \triangleq \dot{y}(0)$

$\ddot{y}_0 \triangleq \ddot{y}(0)$

Example 30

1810

Suppose we have a transfer function:

$$u \to \boxed{H} \to y$$

$$\boxed{H(s) = \left.\frac{Y(s)}{U(s)}\right|_{\text{zero ICs}} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}}$$

then $\Longrightarrow$ the associated differential equation relating y & u is given by:

$$\boxed{\begin{aligned} a_n \overset{(n)}{y} + a_{n-1} \overset{(n-1)}{y} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y \\ = b_m \overset{(m)}{u} + b_{m-1} \overset{(m-1)}{u} + \cdots + b_2 \ddot{u} + b_1 \dot{u} + b_0 u \end{aligned}}$$

Example 30

Initial Value Theorem

Suppose

$$y(t) = y_0 + \dot{y}_0 t + \ddot{y}_0 \frac{t^2}{2!} + \cdots$$

$$\rightarrow \quad Y = \frac{y_0}{s} + \frac{\dot{y}_0}{s^2} + \frac{\ddot{y}_0}{s^3} + \frac{\dddot{y}}{s^4} + \cdots$$

$$sY = y_0 + \frac{\dot{y}_0}{s} + \frac{\ddot{y}_0}{s^2} + \frac{\dddot{y}_0}{s^3} + \cdots$$

$$y_0 = \lim_{s \to \infty} sY(s)$$

$$s[sY - y_0] = \dot{y}_0 + \frac{\ddot{y}_0}{s} + \frac{\dddot{y}_0}{s^2} + \cdots$$

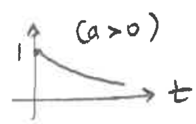
$$\dot{y}_0 = \lim_{s \to \infty} s^2 Y(s) - s y_0$$

$$s\left[s[sY - y_0] - \dot{y}_0\right] = \ddot{y}_0 + \frac{\dddot{y}_0}{s} + \cdots$$

$$\ddot{y}_0 = \lim_{s \to \infty} s^3 Y - s^2 y_0 - s\dot{y}_0$$

$$y_0^{(n)} = \lim_{s \to \infty} s^n Y - s^{n-1} y_0 - \cdots - s^2 y_0^{(n-3)} - s y_0^{(n-2)} - y_0^{(n-1)}$$

Application:

Suppose $y(t) = e^{-at}$ 

$\leftrightarrow \quad Y(s) = \frac{1}{s+a}$

We see $y(0^+) = 1$

Lets use the initial value theorem to get this:

$$y(0^+) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} s\left(\frac{1}{s+a}\right) = 1$$

as expected!

Example 30

Final Value Theorem

1830

Suppose $y \leftrightarrow Y$

& $y_\infty = y(\infty) =$ final value of y

Claim: $$\boxed{y_\infty = \lim_{s\to 0} s\,Y(s)}$$

Pf:

$$Y(s) = \int_0^\infty y(\tau)\, e^{-s\tau}\, d\tau$$

suppose $y \approx y_\infty$ for $t \geq T$

$$= \int_0^T y(\tau)\, e^{-s\tau}\, d\tau + \int_T^\infty y_\infty\, e^{-s\tau}\, d\tau$$

$$= \int_0^T y(\tau)\, e^{-s\tau}\, d\tau + y_\infty \left. \frac{e^{-s\tau}}{s} \right|_\infty^T$$

$$\frac{e^{-sT}}{s} - \frac{e^{-s\infty}}{s}$$

0 (for $s > 0$)

$$s\,Y(s) = s\int_0^T y(\tau)\, e^{-s\tau}\, d\tau + y_\infty \left[e^{-sT} - \cancel{e^{-s\infty}} \right]$$

Now let $s \to 0$

$$\lim_{s\to 0} s\,Y(s) = 0 + y_\infty\,[1 - 0]$$

$$\Longrightarrow \quad y_\infty = \lim_{s\to 0} s\,Y(s)$$

Application:

Suppose $y(t) = 1 - e^{-t}$

$\leftrightarrow \quad Y = \frac{1}{s} - \frac{1}{s+1} = \frac{s+1-s}{s(s+1)}$

$= \frac{1}{s(s+1)}$

We see that $y_\infty = 1$

Lets use the final value Theorem to get this:

$$y_\infty = \lim_{s\to 0} s\,Y(s) = s \left. \frac{1}{s(s+1)} \right|_{s=0} = \left. \frac{1}{s+1} \right|_{s=0} = 1$$

as expected!

Example 30

Find $y(t)$ for the follow $Y(s)$ by forming a partial fraction expansion & taking inverse Laplace Transforms term by term

numerator is set to 1 only for simplicity

$$Y(s) = \frac{1}{s\,(s+1)\,(s-4)\,(s^2-s+1)\,(s^2+2s+2)\,(s^2+100)}$$

poles $= \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ poles $= -1 \pm j1$ poles $= \pm j10$

Here is the partial fraction expansion:

conjugate $= \frac{\bar{D}}{s-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$ conjugate $= \frac{\bar{E}}{s+1+j1}$ conjugate $= \frac{\bar{F}}{s+j10}$

$$Y = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-4} + \left[\frac{D}{s-\frac{1}{2}-j\frac{\sqrt{3}}{2}} + *\right] + \left[\frac{E}{s+1-j1} + *\right] + \left[\frac{F}{s-j10} + *\right]$$

Using our provided transform pairs we can now get the form of $y(t)$:

Note: These are growing (unstable) terms!

$$y(t) = A + Be^{-t} + Ce^{4t} + 2|D|e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle D\right)$$
$$+ 2|E|e^{-t}\cos(t + \angle E)$$
$$+ 2|F|\cos(10t + \angle F)$$

This $= y_{ss}$ since it dominates after a long time

We still need to compute the coefficients: A, B, C, D, E, F

Lets now show that they are all computed using the following formula:

$$\boxed{\text{Coeff} = \lim_{\substack{s \to \text{pole} \\ \text{associated} \\ \text{with coeff}}} (s - \text{pole})\, Y(s)}$$

we have

$$\Rightarrow \quad Y = \cdots + \left[\frac{\text{coeff}}{s - \text{pole}} + * \right] + \cdots$$

To find coeff, multiply both sides by $(s - \text{pole})$

$$(s - \text{pole})\,Y = \text{coeff} + \left[\text{all other terms} \right] (s - \text{pole})$$

Now take $\lim\limits_{s \to \text{pole}}$.

This yields

$$\lim_{s \to \text{pole}} (s - \text{pole})\, Y(s) = \text{coeff} + \cancel{\left[\text{all other terms} \right] (s - \text{pole})}\Big|_{s = \text{pole}} \quad 0$$

or

$$\text{coeff} = \lim_{s \to \text{pole}} (s - \text{pole})\, Y(s)$$

as stated above!

From this, it follows therefore that

$$\boxed{A = \lim_{s \to 0} s\, Y(s)} = \frac{1}{(s+1)(s-4)(s^2 - s + 1)(s^2 + 2s + 2)(s^2 + 100)}\Bigg|_{s=0}$$

$$= \frac{1}{(1)(-4)(1)(2)(100)}$$

$$\boxed{B = \lim_{s \to -1} (s+1)\, Y(s)} = \frac{1}{s\,(s-4)(s^2 - s + 1)(s^2 + 2s + 2)(s^2 + 100)}\Bigg|_{s=-1}$$

$$= \frac{1}{(-1)(-5)(1+1+1)(1-2+2)(1+100)}$$

$$\boxed{C = \lim_{s \to 4} (s-4)\,Y(s)} = \left.\frac{1}{s(s+1)(s^2-s+1)(s^2+2s+2)(s^2+100)}\right|_{s=4}$$

$$= \frac{1}{(4)(5)(16-4+1)(16+8+2)(16+100)}$$

$$\boxed{D = \lim_{s \to \frac{1}{2}+j\frac{\sqrt{3}}{2}} \left(s-\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2}\right) Y(s)} = \left.\frac{1}{s(s+1)(s-4)\left(s-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)(s^2+2s+2)(s^2+100)}\right|_{s=\frac{1}{2}+j\frac{\sqrt{3}}{2} = 1e^{j60°}}$$

$$= \frac{1}{\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(\frac{3}{2}+j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}-4\right)\left(j\sqrt{3}\right)\left(1e^{j120°}+1+j\sqrt{3}+2\right)}$$

$(1e^{j120}+100)$

etc... you can do the rest

$$\boxed{E = \lim_{s \to -1+j1} (s+1-j1)\,Y(s)} = \left.\frac{1}{s(s+1)(s-4)(s^2-s+1)(s+1+j1)(s^2+100)}\right|_{s=-1+j1 = \sqrt{2}e^{j135}}$$

$$= \frac{}{(-1+j1)(j1)(-5+j1)(2e^{j270}+1-j1+1)(j2)(\quad)}$$

$2e^{j270°}+100$

etc... you can do the rest

$$\boxed{F = \lim_{s \to j10} (s-j10)\,Y(s)} = \left.\frac{1}{s(s+1)(s-4)(s^2-s+1)(s^2+2s+2)(s+j10)}\right|_{s=j10}$$

$$= \frac{1}{(j10)(j10+1)(j10-4)(-100-j10+1)(-100+j20+2)(j20)}$$

etc... you can do the rest

Important Note:

> The most import coefficients to compute are always those with critical steady state constants, sinusoids, & growing signals!

Example 30

1870

Inverse Transforms

For each Y, find $y(t)$, y_{ss}, & t_s.

1 $Y(s) = \left[\dfrac{1}{s+1}\right]\dfrac{10}{s}$

$t_s = 5\tau = \dfrac{5}{|\text{Re pole}|} = \dfrac{5}{|-1|} = 5\text{ sec}$

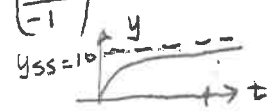
Solution:

$$Y = \frac{A}{s} + \frac{B}{s+1} \Rightarrow y = \underset{y_{ss}}{A} + Be^{-t}$$

stable (decaying exponential)

$$A = \lim_{s\to 0} sY = s\left[\frac{1}{s+1}\right]\frac{10}{s}\Big|_{s=0}$$
$$= 10\left[\frac{1}{s+1}\right]\Big|_{s=0}$$
$$= 10\,(1)$$
$$B = \lim_{s\to -1}(s+1)Y = (s+1)\left[\frac{1}{s+1}\right]\frac{10}{s}\Big|_{s=-1}$$
$$= \left(\frac{10}{-1}\right) = -A$$

$$\Rightarrow y = 10(1)\left[1 - e^{-t}\right]$$



$t_s = 5\text{sec}$

2 $Y(s) = \left[\dfrac{1}{s+1}\right]\left[\dfrac{1}{s^2+1}\right]$

poles $= \pm j1$

$t_s = 5\tau = \dfrac{5}{|\text{Re pole}|} = \dfrac{5}{|-1|} = 5\text{ sec}$

Solution:

$$Y = \frac{A}{s+1} + \frac{B}{s-j1} + * \Rightarrow y = Ae^{-t} + \underset{y_{ss}}{2|B|\cos(t + \angle B)}$$

conjugate $= \dfrac{B}{s+j1}$

$$A = \lim_{s\to -1}(s+1)Y = \frac{1}{s^2+1}\Big|_{s=-1} = \frac{1}{2}$$
$$B = \lim_{s\to j1}(s-j1)Y = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+j1}\right)\Big|_{s=j1}$$
$$= \left(\frac{1}{j1+1}\right)\frac{1}{j2}$$

$\sqrt{2}$, $45°$, 1, 1; $j2$, $2\angle 90°$

$$= \left(\frac{1}{\sqrt{2}e^{j45°}}\right)\left(\frac{1}{2e^{j90°}}\right)$$
$$= \frac{1}{2\sqrt{2}}e^{-j135°}$$

$$\Rightarrow y = \frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - 135°)$$

$45 + 90°$

Note: $\cos(x - 90) = \sin x$

$$= \frac{1}{2}e^{-t} + \underset{y_{ss}}{\frac{1}{\sqrt{2}}\sin(t - 45°)}$$

Example 30

1880

Inverse Transforms

[3] $Y(s) = \left[\frac{1}{s-3} \right] \left[\frac{9}{s} \right]$

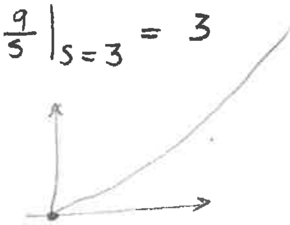
Solution:

$Y = \frac{A}{s} + \frac{B}{s-3} \Longrightarrow y = A + Be^{3t}$ ← yss

unstable (growing exponential!)

$A = \lim_{s \to 0} sY = \left. \frac{9}{s-3} \right|_{s=0} = -3$

$B = \lim_{s \to 3} (s-3)Y = \left. \frac{9}{s} \right|_{s=3} = 3$



$y = 3[-1 + e^{3t}]$

$y_{ss} = Be^{3t} = 3e^{3t}$

↑ unstable (growing exponential)

How do we find a settling time here?

Answer: When is $\left| \frac{y - y_{ss}}{y_{ss}} \right| < 0.1$?

$\Longrightarrow y = -3 + y_{ss}$

$\Longrightarrow y - y_{ss} = -3$; $y_{ss} = 3e^{3t}$

$\Longrightarrow \left| \frac{y - y_{ss}}{y_{ss}} \right| = \left| \frac{-3}{3e^{3t}} \right| = e^{-3t} < 0.1$

$10 < e^{3t}$

$\ln 10 < 3t$

$\frac{\ln 10}{3} < t$

$\Rightarrow \underset{1\%}{t_s} = \frac{\ln 10}{3}$

Example 30

$\boxed{4}$ $Y(s) = \left(\frac{1}{s-1}\right)\left[\frac{100}{s^2+104}\right]$

$\uparrow$ poles $= \pm j100$

solution:

conjugate $= \frac{\bar{B}}{s+j100}$

$$y = \frac{A}{s-1} + \frac{B}{s-j100} + * \Rightarrow y = Ae^{t} + 2|B|\cos(100t + \angle B)$$

unstable (growing exponential)

$\uparrow$ yss

$$A = \lim_{s\to 1}(s-1)Y = \left.\frac{100}{s^2+104}\right|_{s=1} = \frac{100}{1+104} \cong 10^{-2}$$

$$B = \lim_{s\to j100}(s-j100)Y = \left.\left[\frac{1}{s-1}\right]\frac{100}{(s+j100)}\right|_{s=j100}$$

$$= \left(\frac{1}{j100-1}\right)\frac{1}{j2}$$

$$\cong \left(\frac{1}{100e^{j90^\circ}}\right)\left(\frac{1}{2e^{j90^\circ}}\right)$$

$$\cong \frac{1}{200}e^{-j180^\circ}$$

$90+90^\circ$

$$\Rightarrow y \cong 10^{-2}e^{t} + \frac{1}{100}\cos(100t - 180^\circ)$$

$\downarrow$ $\cos(x-90^\circ) = \sin x$

$$\cong 10^{-2}e^{t} + 10^{-2}\sin(100t - 90^\circ)$$

$\uparrow$ yss (unstable growing exponential)

How do we find a setting time here?

Answer:

When is $\left|\frac{y-y_{ss}}{y_{ss}}\right| \leq 0.1$?

$$\Rightarrow \left|\frac{y-y_{ss}}{y_{ss}}\right| = \frac{|10^{-2}\sin(100t-90^\circ)|}{10^{-2}e^{t}}$$

$$\leq \frac{10^{-2}}{10^{-2}}e^{-t} \leq 0.1$$

$$10 \leq e^{t}$$

$$\ln 10 \leq t$$

$$t_{1\%} = \ln 10$$

Example 30

5 $$Y(s) = \left[\frac{1}{s-1}\right]\left[\frac{s^2+s+1}{s(s^2+1)}\right]$$

$\uparrow$ poles $= \pm j1$

Solution:

$$Y = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s-j1} + *$$

conjugate $= \frac{\bar{C}}{s+j1}$

$$y = Ae^{t} + B + 2|C|\cos(t + \angle C)$$

yss

unstable (growing exponential)

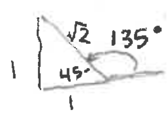
$$A = \lim_{s\to 1}(s-1)Y = \left.\frac{s^2+s+1}{s(s^2+1)}\right|_{s=1} = \frac{1+1+1}{(1)(1+1)} = \frac{3}{2}$$

$$B = \lim_{s\to 0} sY = \left.\left[\frac{1}{s-1}\right]\left[\frac{s^2+s+1}{s^2+1}\right]\right|_{s=0} = (-1)(1) = -1$$

$$C = \lim_{s\to j1}(s-j1)Y = \left.\left[\frac{1}{s-1}\right]\left[\frac{s^2+s+1}{s(s+j1)}\right]\right|_{s=j1}$$

$$= \left[\frac{1}{j1-1}\right]\left[\frac{-1+j1+1}{(j1)(j2)}\right]$$

$$= \left[\frac{1/2}{-1+j1}\right]\left[\frac{1}{j1}\right] = \left(\frac{1/2}{\sqrt{2}e^{j135}}\right)\left(\frac{1}{e^{j90}}\right) = \frac{1}{2\sqrt{2}}e^{-j225}$$



$135 + 90°$

$$y = \frac{3}{2}e^{t} + (-1) + \frac{1}{\sqrt{2}}\cos(t - 225°)$$

$$= \frac{3}{2}e^{t} + (-1) + \frac{1}{\sqrt{2}}\sin(t - 135°) \quad \leftarrow \sin(x) = \cos(x - 90°)$$

yss (unstable growing exponential)

$$t_{1\%} = \ln\left[\frac{20(1+\frac{1}{\sqrt{2}})}{3}\right]$$

How do we find a settling time here? **Answer:** When is $\left|\frac{y-y_{ss}}{y_{ss}}\right| \le 0.1$?

$$\Rightarrow \left|\frac{y-y_{ss}}{y_{ss}}\right| = \left|\frac{-1+\frac{1}{\sqrt{2}}\sin(t-135°)}{\frac{3}{2}e^{t}}\right| \le \frac{1+\frac{1}{\sqrt{2}}}{\frac{3}{2}e^{t}} = \frac{2}{3}\left(1+\frac{1}{\sqrt{2}}\right)e^{-t} \le 0.1$$

$$\Rightarrow \frac{20(1+\frac{1}{\sqrt{2}})}{3} \le e^{t}$$

$$\Rightarrow \ln\left[\frac{20(1+\frac{1}{\sqrt{2}})}{3}\right] \le t$$

Example 30

2000

$\boxed{6}$ $$Y(s) = \left[\frac{100}{(s+1)(s+2)(s+10)}\right]\left[\frac{10}{s}\right]$$

solution:

$$Y = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+10}$$

slowest pole at $s=-1$ $\Rightarrow$ $t_s = 5\tau = \frac{5}{|\text{real pole}|} = \frac{5}{|-1|} = 5 \text{ sec}$

$$y = \underbrace{A}_{y_{ss}} + Be^{-t} + Ce^{-2t} + De^{-10t}$$

$$A = \lim_{s\to 0} sY = \left[\frac{100}{(s+1)(s+2)(s+10)}\right] 10 \Big|_{s=0} = \left[\frac{(100)}{(1)(2)(10)}\right] 10 = (5)(10)$$

$$B = \lim_{s\to -1} (s+1)Y = \left[\frac{100}{(s+2)(s+10)}\right]\frac{10}{s}\Big|_{s=-1} = \left[\frac{100}{(1)(9)}\right]\frac{10}{-1} = -\frac{1000}{9}$$

$$C = \lim_{s\to -2} (s+2)Y = \left[\frac{100}{(s+1)(s+10)}\right]\frac{10}{s}\Big|_{s=-2} = \left[\frac{100}{(-1)(8)}\right]\frac{10}{-2} = \frac{125}{2}$$

$$D = \lim_{s\to -10} (s+10)Y = \left[\frac{100}{(s+1)(s+2)}\right]\frac{10}{s}\Big|_{s=-10} = \left[\frac{100}{(-9)(-8)}\right]\frac{10}{-10} = \frac{25}{18}$$

$$y = \underbrace{10(5)}_{y_{ss}} - \underbrace{\frac{1000}{9}e^{-t}}_{t_s = 5 \text{ sec}} + \frac{125}{2}e^{-2t} + \frac{25}{18}e^{-10t}$$

Example 30

2010

$\boxed{7}$ $Y(s) = \left[\dfrac{250}{(s+7)(s^2+6s+25)}\right]\left[\dfrac{4}{s}\right]$

solution:

$(s+7)$: fast pole

$(s^2+6s+25)$: poles $= -3 \pm j4$ ← slow poles

$t_s = \dfrac{5}{|\text{Re pole}|} = \dfrac{5}{|-3|} = \dfrac{5}{3}$

$Y = \dfrac{A}{s} + \dfrac{B}{s+7} + \dfrac{C}{s+3-j4} + *$ ← conjugate $= \dfrac{\bar{C}}{s+3+j4}$

$y = A + Be^{-7t} + 2|C|\, e^{-3t} \cos(4t + \angle C)$

A ← yss

$A = \lim_{s \to 0} sY = \left[\dfrac{250}{(s+7)(s^2+6s+25)}\right] 4 \Big|_{s=0} = \left[\dfrac{250}{(7)(25)}\right] 4 = 4\left(\dfrac{10}{7}\right)$ ← 10/7

$B = \lim_{s \to -7} (s+7)Y = \left[\dfrac{250}{s^2+6s+25}\right]\dfrac{4}{s}\Big|_{s=-7} = \left[\dfrac{250}{\underbrace{49-42+25}_{32}}\right]\dfrac{4}{-7} = -\dfrac{125}{28}$

$C = \lim_{s \to -3+j4} (s+3-j4)Y = \left[\dfrac{50}{(s+7)(s+3+j4)}\right]\dfrac{4}{s}\Big|_{s=-3+j4}$

$= \left[\dfrac{50}{\underbrace{(4+j4)}_{4\sqrt{2}e^{j45}}\underbrace{(j8)}_{8e^{j90^\circ}}}\right]\dfrac{4}{-3+j4}$

4, 4; 4, 3, 5, 53°, 127°

$= \left[\dfrac{50}{(4\sqrt{2}e^{j45^\circ})(8e^{j90^\circ})}\right]\dfrac{4}{5e^{j127^\circ}}$

$= \dfrac{2.5}{2\sqrt{2}}\, e^{-j(\underbrace{45+90+127^\circ}_{262^\circ})}$

$y = \underbrace{4\left(\tfrac{10}{7}\right)}_{yss} - \dfrac{125}{28}e^{-7t} + \dfrac{2.5}{\sqrt{2}}\, e^{-3t}\cos(4t - 262^\circ)$

$t_s = \dfrac{5}{3}$

Example 30

$\boxed{8}$ $$Y(s) = \left[\frac{1}{s^2+s+1}\right]\left[\frac{10s^2+s+10}{s(s^2+1)}\right]$$

poles $= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$; poles $= \pm j1$

$t_s = \frac{5}{|\text{Re pole}|} = \frac{5}{|-\frac{1}{2}|} = 10$ sec

Solution:

$$Y = \frac{A}{s} + \left[\frac{B}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}} + *\right] + \left[\frac{C}{s-j1} + *\right]$$

yss

$$y = A + 2|B|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle B\right) + 2|C|\cos(t + \angle C)$$

decaying exponential: ts = 10sec

$$A = \lim_{s\to 0} sY = \left[\frac{1}{s^2+s+1}\right]\left[\frac{10s^2+s+10}{s^2+1}\right]\Bigg|_{s=0} = (1)\left(\frac{10}{1}\right) = (10)(1)$$

$$B = \lim_{s\to -\frac{1}{2}+j\frac{\sqrt{3}}{2}} \left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)Y = \left[\frac{1}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}}\right]\left[\frac{10s^2+s+10}{s^2+1}\right]\Bigg|_{s=-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$= 1e^{j120°}$

$\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, 60, 120°

240°, 60°, 10, $-5+j5\sqrt{3}$

$$= \left[\frac{1}{j\sqrt{3}}\right]\left[\frac{10e^{j240} - \frac{1}{2}+j\frac{\sqrt{3}}{2}+10}{e^{j240}+1}\right]$$

$\sqrt{3}e^{j90°}$; $\frac{1}{2}-j\frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}$, 1/2, 60

$1e^{j240°} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

$5.5\sqrt{3}$, 4.5

$$= \left[\frac{1}{\sqrt{3}e^{j90}}\right]\left[\frac{4.5+j5.5\sqrt{3}}{\frac{1}{2}-j\frac{\sqrt{3}}{2}}\right]$$

$1e^{-j60°}$; $\frac{1}{2}$, 60, $\frac{\sqrt{3}}{2}$

$$= \frac{\sqrt{(4.5)^2+(5.5\sqrt{3})^2}}{(\sqrt{3})(1)} e^{j\left(\tan^{-1}\left(\frac{5.5\sqrt{3}}{4.5}\right) - 90° - (-60°)\right)}$$

$$C = \lim_{s\to j1}(s-j1)Y$$

$$= \left[\frac{1}{s^2+s+1}\right]\left[\frac{10s^2+s+10}{s(s+j1)}\right]\Bigg|_{s=j1}$$

$$= \left(\frac{1}{-1+j1+1}\right)\left(\frac{-10+j1+10}{(j1)(j2)}\right)$$

$$= \left(\frac{1}{\cancel{j1}}\right)\left(\frac{\cancel{j1}}{(j1)(j2)}\right)$$

$$= \frac{1}{(1)(2)e^{j(90+90)}}$$

$$= \frac{1}{2}e^{-j180°}$$

Note: A & C are much more important to compute than B because A & C are associated with yss & B is associated with a decaying exponential with ts = 10 sec.

yss ; $-90-90$

$$y = (10)(1) + 1\cos(t-180°) + 2|B|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle B\right)$$

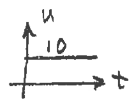
$= \sin(t-90°)$; ts = 10 sec

Problem 30

For each of the following systems H, find $y(t)$, y_{ss}, t_s

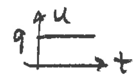
$u \rightarrow \boxed{H} \rightarrow y$

$\boxed{1}$ $H(s) = \dfrac{1}{s+1}$



$\boxed{2}$ $H(s) = \dfrac{1}{s+1}$ $\quad u(t) = \sin t$

$\boxed{3}$ $H(s) = \dfrac{1}{s-3}$

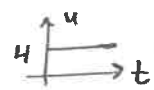


$\boxed{4}$ $H(s) = \dfrac{1}{s-1}$ $\quad u(t) = \sin 100t$

$\boxed{5}$ $H(s) = \dfrac{1}{s-1}$ $\quad u(t) = 1 + \sin t$

$\boxed{6}$ $H(s) = \dfrac{100}{(s+1)(s+2)(s+3)}$

$\boxed{7}$ $H(s) = \left[\dfrac{250}{(s+7)(s^2+6s+25)}\right]$



$\boxed{8}$ $H(s) = \left[\dfrac{1}{s^2+s+1}\right]$ $\quad u(t) = 10 + \sin t$

$\boxed{9}$ $H(s) = \dfrac{s}{s+1}$

$\boxed{10}$ $H(s) = \dfrac{s^2+9}{s^2+s+1}$ $\quad u(t) = \cos(3t - 60^\circ)$

Hint:
See some $\boxed{1}$ - $\boxed{8}$ in Example 30.
(pages 1870 - 2020)