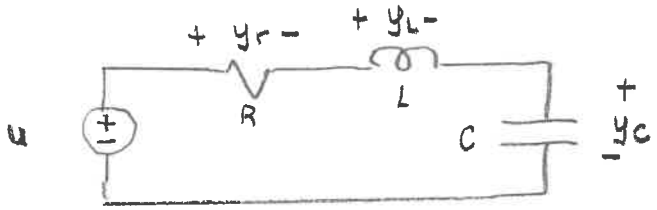
Example 33

Resonant RLC Circuits



Hint: Voltage divide with
$Z_C = \frac{1}{sC}$
$Z_L = sL$
$Z_R = R$

1 Compute transfer function H_C from u to y_C.

$$H_C = \frac{y_C}{u} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

2 Compute the transfer function H_R from u to y_R

$$H_R = \frac{y_R}{u} = \frac{R}{R + sL + \frac{1}{sC}} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

3 Compute the transfer function H_L from u to y_L

$$H_L = \frac{y_L}{u} = \frac{sL}{R + sL + \frac{1}{sC}} = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Note: $H_L + H_R + H_C = 1$

Why? $\frac{y_L}{u} + \frac{y_R}{u} + \frac{y_C}{u} = 1$

Why? By KVL = $u = y_R + y_L + y_C$

4 Compute the transfer function H_{LC} from u to $y_{LC} = y_L + y_C$.

$$H_{LC} = \frac{y_{LC}}{u} = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L} + \frac{1}{LC}}$$

Note: $H_R + H_{LC} = 1$

since $\frac{y_R}{u} + \frac{y_{LC}}{u} = \frac{y_R}{u} + \frac{y_L + y_C}{u} = \frac{y_R}{u} + \frac{y_L}{u} + \frac{y_C}{u} = 1$

Example 33

[5] What is the differential equation associated with y_C, y_R, y_L, y_{LC}

$$H_C = \frac{y_C}{u} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) y_C = \frac{1}{LC} u$$

$$\Rightarrow \ddot{y}_C + \frac{R}{L}\dot{y}_C + \frac{1}{LC} y_C = \frac{1}{LC} u$$

$$H_R = \frac{y_R}{u} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) y_R = \left(\frac{R}{L}s\right) u$$

$$\Rightarrow \ddot{y}_R + \frac{R}{L}\dot{y}_R + \frac{1}{LC} y_R = \frac{R}{L}\dot{u}$$

$$H_L = \frac{y_L}{u} = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) y_L = (s^2) u$$

$$\Rightarrow \ddot{y}_L + \frac{R}{L}\dot{y}_L + \frac{1}{LC} y_L = \ddot{u}$$

$$H_{LC} = \frac{y_{LC}}{u} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) y_{LC} = \left(s^2 + \frac{1}{LC}\right) u$$

$$\Rightarrow \ddot{y}_{LC} + \frac{R}{L}\dot{y}_{LC} + \frac{1}{LC} y_{LC} = \ddot{u} + \frac{1}{LC} u$$

Recall:

$$\boxed{\begin{array}{l} y \leftrightarrow Y \\ \dot{y} \leftrightarrow sY \\ \ddot{y} \leftrightarrow s^2 Y \end{array}}$$

(assuming zero initial conditions)

[6] What is the system characteristic equation (polynomial) Φ ?

$$\Phi = \text{denominator of any system transfer function } H$$

$$= s^2 + \frac{R}{L}s + \frac{1}{LC}$$

[6] What are the system poles (characteristic roots) ?

$$\Phi(s) = s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\Rightarrow s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2(1)}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Recall:

Quadratic Formula:

$$as^2 + bs + c = 0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 33

$\boxed{7}$ What is the system discriminant?

discriminant = quantity underneath radical $\sqrt{(\cdot)}$

$$= \left(\frac{R}{2L}\right)^2 - \frac{1}{LC}$$

This is important because it determines the nature of the roots.

$\boxed{8}$ What can be said about the system poles (roots) when discriminant = 0? For what R is discriminant zero?

When discriminant = 0 $\Rightarrow$ system poles (roots) are:

$$s_{1,2} = -\frac{R}{L}, -\frac{R}{L}$$

Note:
In this case, system 5 time constant or 1% settling time is:

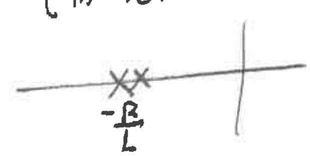
$$t_s = \frac{5}{|\text{Re poles}|} = \frac{5}{\left|-\frac{R}{L}\right|} = 5\left(\frac{L}{R}\right)$$

$\Rightarrow$ They are therefore

real

repeated

stable (in left half plane)



Critical Value of R:

$$\text{discriminant} = \left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0 \iff \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

$$\iff \frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \boxed{R = R_c \triangleq 2\sqrt{\frac{L}{C}}}$$

For this value of R, we say that the system is $\boxed{\text{Critically Damped}}$

(a "critical" level of resistance (damping))

9 What can be said about the system poles (roots) when disc < 0?

For what R is disc < 0?

discriminant < 0 $\Longrightarrow$ system poles (roots) are

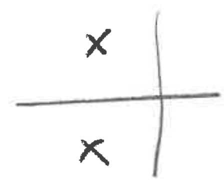
$$s_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$\Longrightarrow$ They are therefore

complex

conjugates

stable (in left half plane)



Note:
In this case, system 5 time constant or 1% settling time is:

$$t_s = \frac{5}{|\text{Re poles}|} = \frac{5}{\left|-\frac{R}{2L}\right|} = \frac{5}{\frac{R}{2L}} = 10\left(\frac{L}{R}\right)$$

disc < 0 $\Longleftrightarrow$ $\boxed{R < R_c \triangleq 2\sqrt{\frac{L}{C}}}$

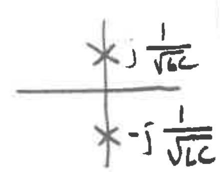
For these values of R, we say that the system is $\boxed{\text{Underdamped}}$

(a "low" level of resistance (damping))

Note: If $R = 0$, then the system poles (roots) are:

$$s_{1,2} = \pm j\frac{1}{\sqrt{LC}}$$

i.e. they are purely imaginary conjugates marginally stable



We shall assume $R > 0$ so that system is stable!!!

10. What can be said about the system poles (roots) when disc > 0?

For what R is disc > 0 ?

disc > 0 $\Longrightarrow$ system poles (roots) are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$\Longrightarrow$ They are therefore

real

distinct (different)

stable (in left half plane)

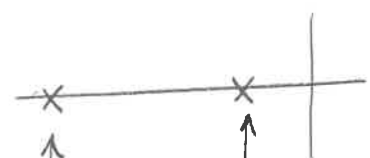
Note:

In this case,

system 5 time constant or 1% settling time is:

$$t_s = \frac{5}{|\mathrm{Re\ slow\ pole}|} = \frac{5}{|s_2|}$$

$$= \frac{5}{\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}}$$



$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

(slow pole)

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

(fast pole)

slow pole determines t_s !!!

disc > 0 $\Longleftrightarrow$ $\boxed{R > R_c \triangleq 2\sqrt{\frac{L}{C}}}$

For these values of R, we say that the system is

$\boxed{\text{Overdamped}}$

(a "high" level of resistance (damping))

3200

$\boxed{11}$ Assuming stability $(R>0)$ for the transfer function H, determine y_{ss} when $u(t) = A + B \underset{\cos}{\sin}(\omega_0 t + \theta)$

steady state output y

Answer:

$$\boxed{\begin{array}{c} u \longrightarrow \underset{\text{stable}}{\boxed{H}} \longrightarrow y \\[1em] u = A + B \underset{\cos}{\sin}(\omega_0 t + \theta) \\[1em] \Longrightarrow \quad y_{ss} = A\,H(0) + B\,|H(j\omega_0)|\,\underset{\cos}{\sin}\left(\omega_0 t + \theta + \angle H(j\omega_0)\right) \end{array}}$$

Note: The above very concisely summarizes all of the work in our texts' Chapter 9!

This is why the following are so EXTREMELY IMPORTANT!

a) transfer function H

b) magnitude response $|H(j\omega)|$

c) phase response $\angle H(j\omega)$

12 From now on, we assume that our system is Very Underdamped; i.e.

$$\boxed{R << R_c \triangleq 2\sqrt{\frac{L}{C}}}$$

(little resistance or damping)

$$H_C = \frac{1}{s^2 + \frac{1}{LC}}$$

$$H_R = 0$$

$$H_L = \frac{s^2}{s^2 + \frac{1}{LC}}$$

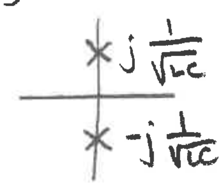
$$H_{LC} = H_L$$

Note:

For $R = 0$, the system poles are

$$s_{1,2} = \pm j \frac{1}{\sqrt{LC}}$$

i.e. they are purely imaginary conjugates marginally stable



Lets use the following numbers moving forward:

$R = 20\,\Omega$

$L = 10\,mH = (10)(10^{-3})\,H = 10^{-2}\,H$

$C = 1\,\mu F = 10^{-6}\,F$

In such a case:

$$\frac{R}{L} = \frac{20}{10^{-2}} = 2000$$

$$\frac{1}{LC} = \frac{1}{(10^{-2})(10^{-6})} = 10^8$$

Example 33 | 3220

13 Determine the system poles -

$$\Phi(s) = s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 2000s + 10^8 = 0$$

$$\Rightarrow s_{1,2} = \frac{-2000 \pm \sqrt{(2\times10^3)^2 - 4(1)(10^8)}}{2(1)}$$

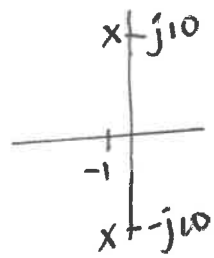
$$= -1000 \pm \sqrt{\tfrac{1}{4}(4)(10^6) - (10^8)}$$

$$= -1000 \pm \sqrt{10^6(1-10^2)}$$

$$= 1000 \pm 1000\sqrt{-99}$$

$$= 1000(-1 \pm j\sqrt{99})$$

$$\cong 1000(-1 \pm j10)$$



14 Lets compute each of the transfer functions H & plot $|H(j\omega)|$ & $\underline{/H(j\omega)}$.

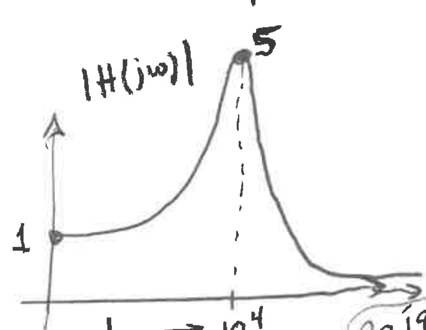
$\boxed{H_c}$

$$H_c(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{10^8}{s^2 + 2000s + 10^8}$$

Resonant Low Pass System

$$H_c(j\omega) = \frac{10^8}{10^8 - \omega^2 + j2000\omega}$$

$$|H_c(j\omega)| = \frac{10^8}{\sqrt{(10^8-\omega^2)^2 + (2000\omega)^2}}$$



$\omega_{resonant} = \frac{1}{\sqrt{LC}} \longrightarrow 10^4$

Note:

$H_c \approx 1$ small s

$H_c \approx \frac{1}{s^2}$ large s

Near $s = j10^4$

$H_c \cong 5e^{-j90°}$

Note: $H(0) = 1 = 1e^{j0°}$ $|H(j\infty)| = 0$

$$H(j10^4) = \frac{10^8}{\cancel{-10^8} + j2000\times10^4 + \cancel{10^8}} = \frac{10^8}{j2\times10^7} = \frac{10}{j2} = 5e^{-j90°}$$

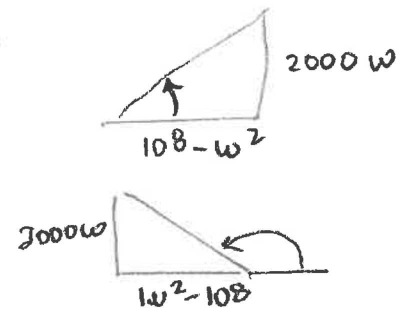
$2e^{j90°}$

Example 33

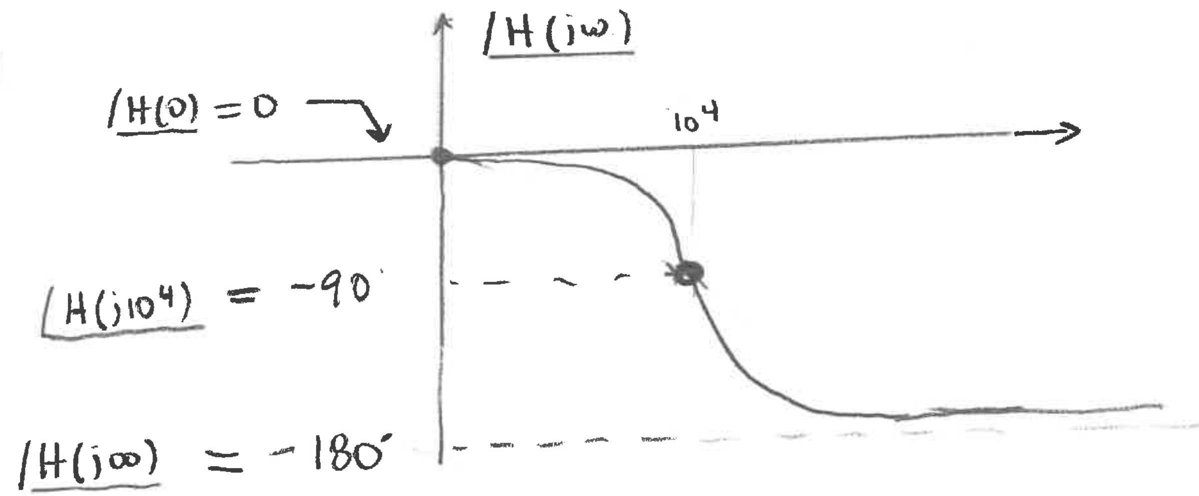
$$\angle H_c(j\omega) = \angle \text{top} - \angle \text{bottom}$$

$$= \angle 10^8 - \angle 10^8 - \omega^2 + j\,2000\,\omega \qquad 10^8 > \omega^2 \;\Rightarrow\; 10^4 > \omega \geq 0$$

$$= 0 - \begin{cases} \text{(triangle 1)} & \text{if } 10^4 > \omega \geq 0 \\ \text{(triangle 2)} & \text{if } 10^4 < \omega < \infty \end{cases}$$



$$= \begin{cases} -\tan^{-1}\left(\dfrac{2000\omega}{10^8-\omega^2}\right) & \text{if } \omega \in [0, 10^4) \\ -\left(180 - \tan^{-1}\left(\dfrac{2000\omega}{\omega^2-10^8}\right)\right) & \text{if } \omega \in [10^4, \infty) \end{cases}$$



H_R

$$H_R(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{2000\,s}{s^2 + 2000\,s + 10^8}$$

Note:

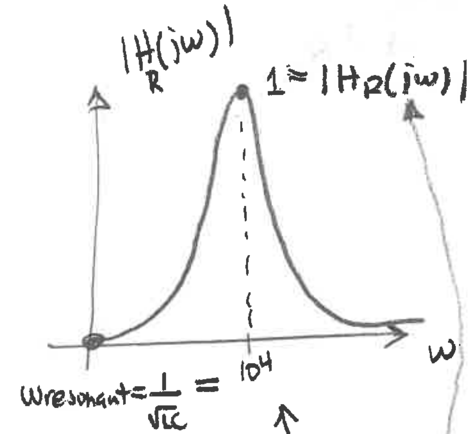
$H_R \cong \frac{2000\,s}{10^8}$ small s

$H_R \cong 1$ near $s \cong j10^4$

$H_R \cong \frac{2000}{s}$ large s

$$H_R(j\omega) = \frac{j\,2000\,\omega}{10^8 - \omega^2 + j\,2000\,\omega}$$

$$|H_R(j\omega)| = \frac{2000\,\omega}{\sqrt{(10^8-\omega^2)^2 + (2000\,\omega)^2}}$$



Resonant Bandpass filter

Note: $H(0) = 0 = 0e^{j0^\circ}$

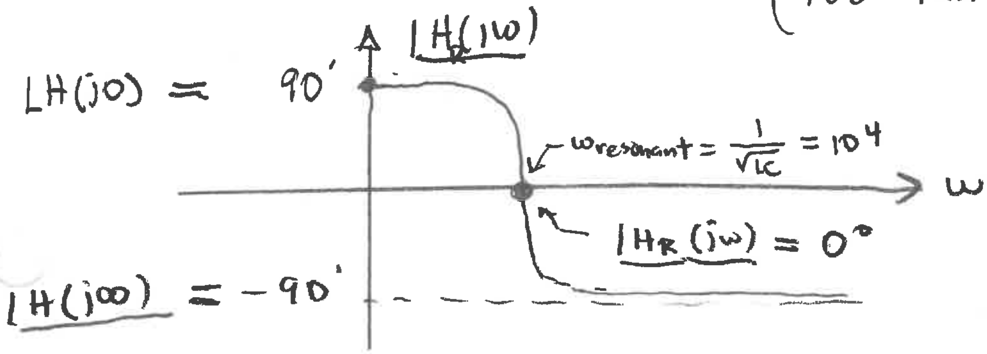
$|H(j\infty)| = 0$

$$H(j10^4) = \frac{2000\,j10^4}{\cancel{-10^8} + j(2000)(10^4) + \cancel{10^8}} = 1 = 1e^{j0^\circ}$$

$$\angle H_R(j\omega) = \angle top - \angle bottom$$

$$= \angle j2000\omega - \angle 10^8 - \omega^2 + j2000\omega$$

$$= 90^\circ - \begin{cases} \tan^{-1}\left(\frac{2000\,\omega}{10^8-\omega^2}\right) & \text{if } \omega \in [0, 10^4) \\ 180 - \tan^{-1}\left(\frac{2000\,\omega}{\omega^2 - 10^8}\right) & \text{if } \omega \in [10^4, \infty) \end{cases}$$



Example 33

$\boxed{H_L}$

$$H_L(s) = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{s^2}{s^2 + 2000s + 10^8}$$

$H_L(s) \cong \frac{s^2}{10^8}$ s small

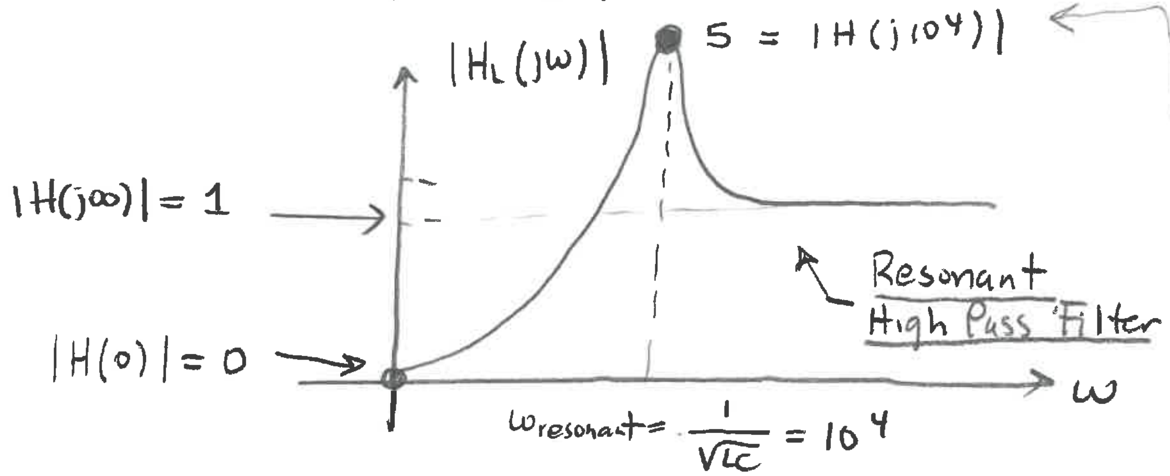
$H_L(s) \cong \frac{s}{2000} = 5e^{j90°}$ near $s \cong j10^4$

$H_L(s) \cong 1$ large s

$$H_L(j\omega) = \frac{(j\omega)^2}{10^8 - \omega^2 + j2000\omega}$$

$(\omega e^{j90°})^2 = \omega^2 e^{j180°}$

$$|H_L(j\omega)| = \frac{\omega^2}{\sqrt{(10^8 - \omega^2)^2 + (2000\omega)^2}}$$



<u>Note:</u>

$H(0) = 0 = 0e^{j0°}$

$H(j\infty) = 1 = 1e^{j0°}$

$$H(j10^4) = \frac{(j10^4)^2}{\cancel{-10^8} + j(2000)(10^4) \cancel{+10^8}}$$

$$= \frac{j10^4}{2000}$$

$$= j5 = 5e^{j90°}$$

Example 33

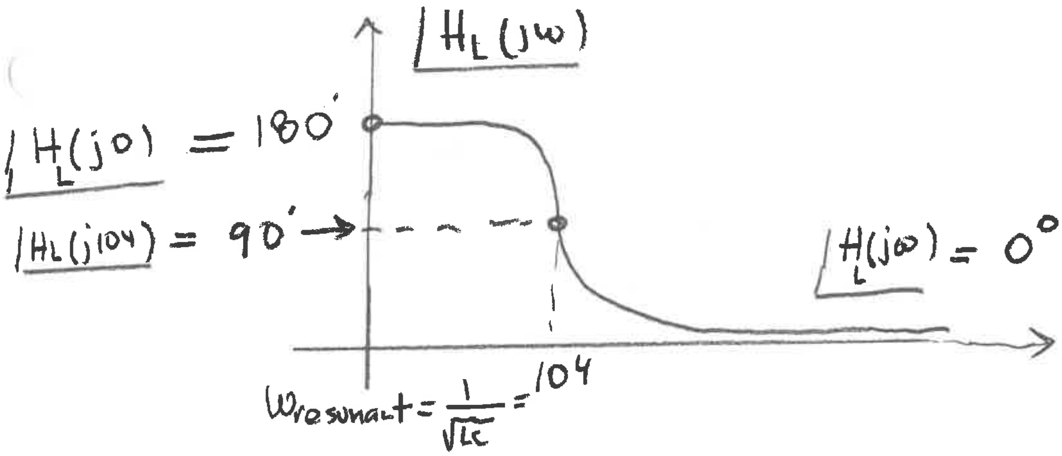
$$\underline{\angle H_L(j\omega)} = \angle \text{top} - \angle \text{bottom}$$

$$= \angle \overset{\omega^2 e^{j180^\circ}}{(j\omega)^2} - \angle\left(10^8 - \omega^2 + j2000\omega\right)$$

$$(j\omega)^2 = (\omega e^{j90^\circ})^2 = \omega^2 e^{j180^\circ}$$

$$= 180^\circ - \begin{cases} \operatorname{Tan}^{-1}\left(\dfrac{2000\omega}{10^8-\omega^2}\right) & \omega \in [0, 10^4) \\ 180 - \operatorname{Tan}^{-1}\left(\dfrac{2000\omega}{\omega^2-10^8}\right) & \omega \in [10^4, \infty) \end{cases}$$

$$= \begin{cases} 180 - \operatorname{Tan}^{-1}\left(\dfrac{2000\omega}{10^8-\omega^2}\right) & \omega \in [0, 10^4) \\ \operatorname{Tan}^{-1}\left(\dfrac{2000\omega}{\omega^2-10^8}\right) & \omega \in [10^4, \infty) \end{cases}$$



Example 33

3270

H_{LC}

$$H_{LC}(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{s^2 + 10^8}{s^2 + 2000s + 10^8}$$

$H_{LC}(s) \cong 1$ s small

$H_{LC}(s) \cong \frac{s^2 + 10^8}{2000s}$ $s \cong j\,10^4$

$H_{LC}(s) \cong 1$ s large

$$H_{LC}(j\omega) = \frac{10^8 - \omega^2}{10^8 - \omega^2 + j\,2000\,\omega}$$

$$|H_{LC}(\omega)| = \frac{|\text{top}|}{|\text{bottom}|}$$

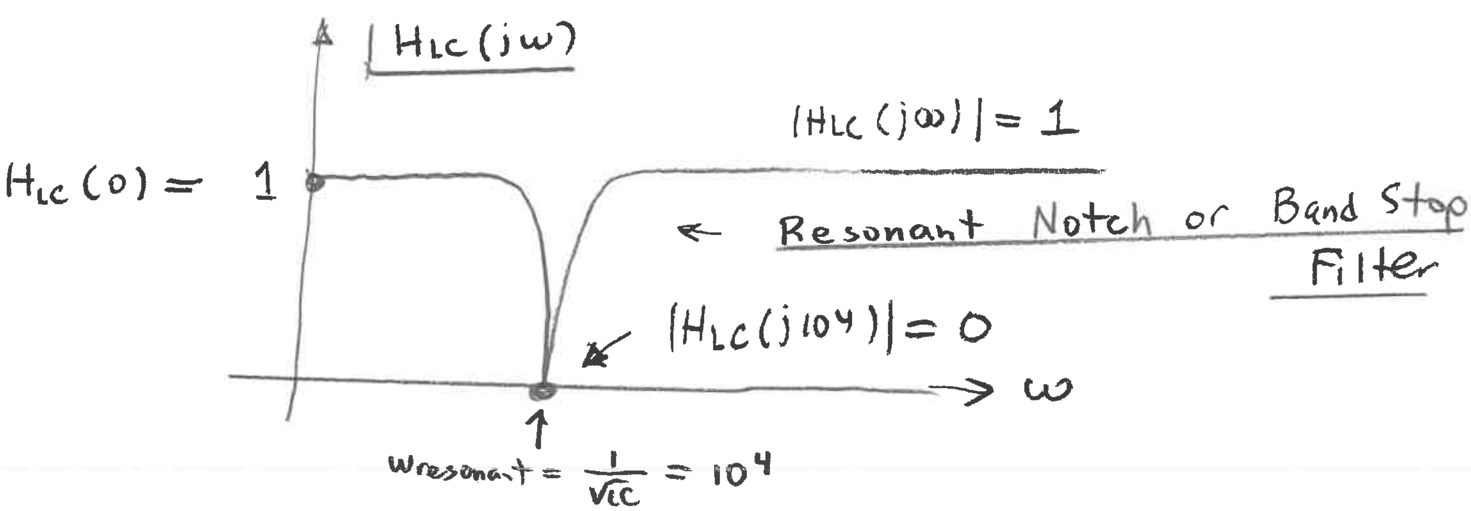
$$= \frac{|10^8 - \omega^2|}{\sqrt{(10^8 - \omega^2)^2 + (2000\,\omega)^2}}$$

<u>Note:</u> $H_{LC}(0) = 1 = 1\,e^{j0^\circ}$

$H_{LC}(j10^4) = 0 = 0\,e^{j0^\circ}$

since numerator of H_{LC} has zeros at $\pm j \frac{1}{\sqrt{LC}} = \pm j\,10^4$

$H_{LC}(j\infty) = 1 = 1\,e^{j0^\circ}$



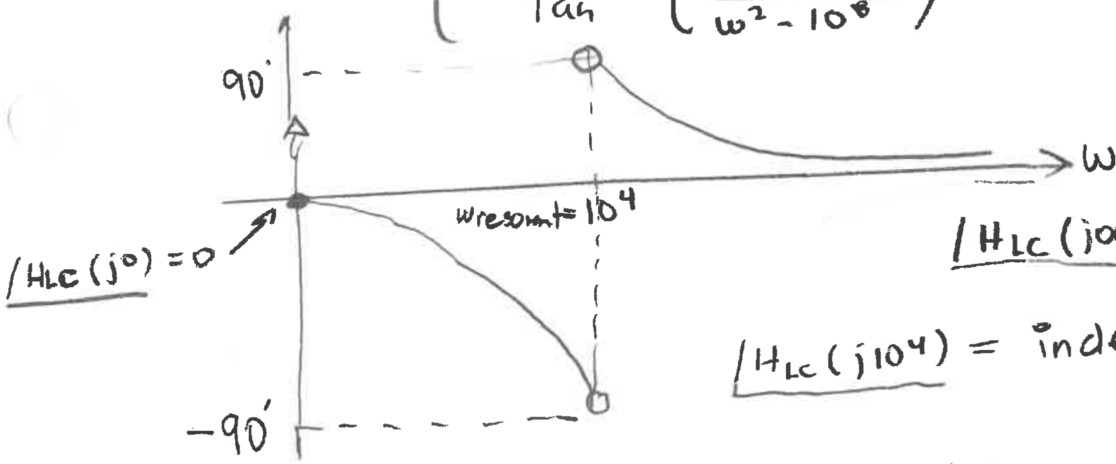
Example 33 | 3280

$$\angle H_{LC}(j\omega) = \angle top - \angle bottom$$

$$= \angle 10^8 - \omega^2 - \angle 10^8 - \omega^2 + j2000\omega$$

$$= \begin{cases} 0^\circ - \tan^{-1}\left(\frac{2000\omega}{10^8-\omega^2}\right) & \text{if } \omega \in [0, 10^4) \\ 180^\circ - \left(180 - \tan^{-1}\left(\frac{2000\omega}{\omega^2-10^8}\right)\right) & \text{if } \omega \in [10^4, \infty) \end{cases}$$

$$= \begin{cases} -\tan^{-1}\left(\frac{2000\omega}{10^8-\omega^2}\right) & \omega \in [0, 10^4) \\ \tan^{-1}\left(\frac{2000\omega}{\omega^2-10^8}\right) & \omega \in [10^4, \infty) \end{cases}$$



$\angle H_{LC}(j\infty) = 0$

$\angle H_{LC}(j10^4)$ = indeterminate !

Note: $H_{LC}(s) \cong 1 = 1e^{j0}$ s small

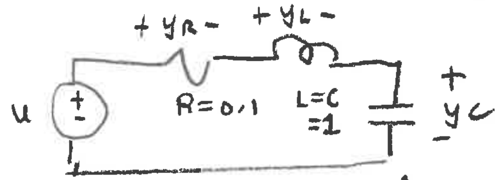
$H_{LC}(s) \cong 1 = 1e^{j0}$ s large

$$H_L(s) \cong \frac{s^2 + 10^8}{2000s} \qquad s \cong j\frac{1}{\sqrt{LC}}$$

$$H_L(j\omega) = \begin{cases} \frac{10^8 - \omega^2}{2000\omega e^{j90^\circ}} = \left(\frac{10^8-\omega^2}{2000\omega}\right)e^{-j90^\circ} & \omega \in [10^{4-}, 10^4) \\ \frac{(\omega^2 - 10^8)e^{j180^\circ}}{2000\omega e^{j90}} = \left(\frac{\omega^2-10^8}{2000\omega}\right)e^{j90^\circ} & \omega \in (10^4, 10^{4+}] \end{cases}$$

Problem 33 | 3290

Consider the RLC circuit



$y_{LC} \triangleq y_L + y_C$

1. Compute $H(s)$, $H(j\omega)$, $|H(j\omega)|$, $\angle H(j\omega)$ for each of the transfer functions H_C, H_R, H_L, H_{LC}

2. Plot $|H(j\omega)|$ & $\angle H(j\omega)$ for each transfer function

Note: clearly indicate what the resonant frequency is in each plot!

Example 34 | Average Power & RMS Values

3300

Consider a device:

$i(t) = |I| \cos(\omega_0 t + \theta_I)$

$+$

$v(t) = |V| \cos(\omega_0 t + \theta_V)$

$-$

This can be any device → --- R, L, C, voltage/current source, ... (or an entire CITY!!!)

Using the trig identity $\boxed{\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)}$

yields:

$$p(t) = v(t)\, i(t)$$

$$= \left[|V|\cos(\omega_0 t + \theta_V)\right]\left[|I|\cos(\omega_0 t + \theta_I)\right]$$

$$= \underbrace{\frac{1}{2}|V||I|\cos(2\omega_0 t + \theta_V + \theta_I)}_{\text{sinusoid with frequency } 2\omega_0 \text{ \& period } \frac{2\pi}{2\omega_0} = \frac{\pi}{\omega_0}} + \underbrace{\frac{1}{2}|V||I|\cos(\theta_V - \theta_I)}_{\text{constant}}$$

If we define $\langle f \rangle \triangleq \frac{1}{T}\int_T f(x)\,dx$

↑ average of a periodic function f with period f

Taking $\langle p \rangle$ yields:

zero!

$$\langle p \rangle = \langle \tfrac{1}{2}|V||I|\cos(2\omega_0 t + \theta_V + \theta_I) \rangle$$

$$+ \langle \tfrac{1}{2}|V||I|\cos(\theta_V - \theta_I) \rangle$$

$$= \tfrac{1}{2}|V||I|\cos(\theta_V - \theta_I)$$

Summary:

$i(t) = |I| \cos(\omega_0 t + \theta_I)$

$+$

$v(t) = |V| \cos(\omega_0 t + \theta_V)$

$-$

$$p(t) = v(t)i(t) = \tfrac{1}{2}|V||I| \{ \cos(2\omega_0 t + \theta_V + \theta_I) + \cos(\theta_V - \theta_I) \}$$

$$\langle p \rangle = \tfrac{1}{2}|V||I| \cos(\theta_V - \theta_I)$$

Lets examine $\langle p \rangle$ for a resistor, inductor, capacitor:
(same v & i above!!!)

Resistor

$v = Ri \Rightarrow |V| = R|I| \qquad \theta_V = \theta_I \quad (\cos(\theta_V - \theta_I) = 1)$

$$\Rightarrow \boxed{\langle p_R \rangle = \tfrac{1}{2}|V||I| = \tfrac{1}{2}R|I|^2 = \tfrac{1}{2}\frac{|V|^2}{R}}$$

Inductor

$v = L\frac{di}{dt} \Rightarrow \frac{V}{I} = sL\Big|_{s=j\omega} = j\omega L$

$\Rightarrow \theta_V - \theta_I = 90'$

$\cos 90 = 0 \Rightarrow \boxed{\langle p_L \rangle = 0}$

Capacitor

$i = C\frac{dv}{dt} \Rightarrow \frac{V}{I} = \frac{1}{sL}\Big|_{s=j\omega L} = \frac{1}{j\omega L}$

$\Rightarrow \theta_V - \theta_I = -90°$

$\Rightarrow \boxed{\langle p_C \rangle = 0}$

Example 34

root mean square

RMS Values

Consider $f(t) = A\cos(\overset{\text{(angular) frequency}}{\omega_0}t + \theta)$

$T = \frac{2\pi}{\omega_0}$ (period)

We now compute:

$$f_{rms} = \sqrt{\frac{1}{T}\int_{\underset{\text{any period}}{T}} f^2(t)\,dt}$$

$$f_{rms}^2 = \frac{1}{T}\int_T f^2(t)\,dt$$

$$= \frac{\omega_0}{2\pi}\int_x^{x+\frac{2\pi}{\omega_0}} [A\cos(\omega_0 t+\theta)]^2\,dt$$

$$\cos^2 x = \tfrac{1}{2}[\cos 2x + 1]$$

$$= \frac{\omega_0}{2\pi}\int_x^{x+\frac{2\pi}{\omega_0}} \frac{A^2}{2}[\cos(2\omega_0 t+2\theta) + 1]\,dt$$

$$= \frac{\omega_0}{2\pi}\frac{A^2}{2}\int_x^{x+\frac{2\pi}{\omega_0}} \cos(2\omega_0 t+2\theta)\,dt + \frac{\omega_0}{2\pi}\frac{A^2}{2}\int_x^{x+\frac{2\pi}{\omega_0}} dt$$

we'll show this = zero below

$$= \frac{\omega_0}{2\pi}\frac{A^2}{2}\left(\frac{2\pi}{\omega_0}\right)$$

$$= \frac{A^2}{2}$$

$$\Longrightarrow \boxed{f_{rms} = \frac{A}{\sqrt{2}} \quad \text{for } f(t) = A\cos(\omega_0 t+\theta)}$$

Example 34

Now lets show that

$$\int_x^{x+\frac{2\pi}{\omega_0}} \cos(2\omega_0 t + 2\theta)\, dt = 0$$

solution:

$$\int_x^{x+\frac{2\pi}{\omega_0}} \cos(2\omega_0 t + 2\theta)\, dt = \left.\frac{\sin(2\omega_0 t + 2\theta)}{2\omega_0}\right|_x^{x+\frac{2\pi}{\omega_0}}$$

$$= \frac{1}{2\omega_0}\left[\sin\left(2\omega_0\left(x+\frac{2\pi}{\omega_0}\right)+2\theta\right) - \sin(2\omega_0 x + 2\theta)\right]$$

$$= \frac{1}{2\omega_0}\left[\sin(2\omega_0 x + 2\theta + 4\pi) - \sin(2\omega_0 x + 2\theta)\right]$$

$\sin(x + 2\pi n) = \sin x$

$$= \frac{1}{2\omega_0}\left[\sin(2\omega_0 x + 2\theta) - \sin(2\omega_0 x + 2\theta)\right]$$

$$= 0$$

as we wanted to show!

Why should anyone care about this RMS value stuff?

For a resistor

$$\boxed{\langle P_R \rangle = \frac{1}{2}|V||I| = \frac{1}{2}R|I|^2 = \frac{1}{2}\frac{|V|^2}{R}}$$

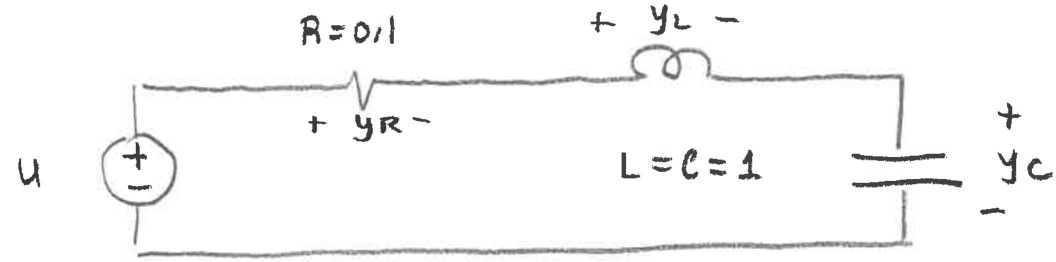
Noting that $V_{rms} = \frac{V}{\sqrt{2}}$; $I_{rms} = \frac{I}{\sqrt{2}}$, the above becomes

$$\boxed{\langle P_R \rangle = |V_{rms}||I_{rms}| = R|I_{rms}|^2 = \frac{|V_{rms}|^2}{R}}$$

Now $\langle P_R \rangle$ formulae look just like ohm's law for a resistor !!!

Problem 34

For the circuit in Problem 33:



compute $\langle P_R \rangle$, $\langle P_L \rangle$, $\langle P_C \rangle$

when $u(t) = A \sin(\omega_0 t + \theta)$

& ω_0 = resonant frequency of the circuit!

Hint: What do H_R, H_L, H_C look like near $s = j\omega_0$?

That will permit you to compute y_{Rss}, y_{Lss}, y_{Css}

Now you can find the steady state current passing through R, L, & C.

i_{ss}

Now you can compute $\langle P_R \rangle$, $\langle P_L \rangle$, $\langle P_C \rangle$

This is the important one! These are trivial!