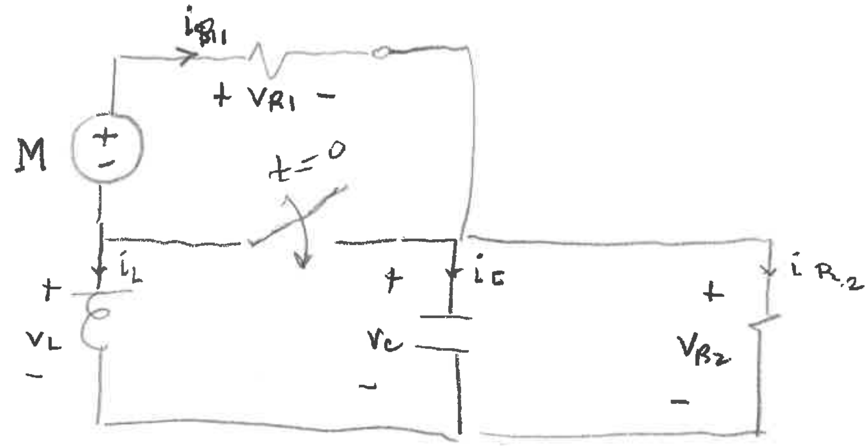
# Example 32 (Switching RLC Circuit)

3010

A

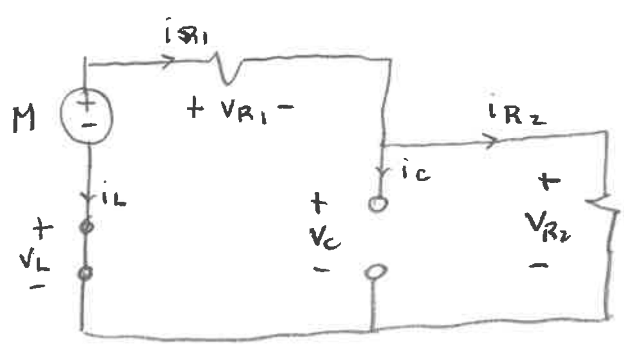
Assume $M>0$




(All $R=L=C=1$)

1. Find $i_{R_1}, i_L, i_C, i_{R_2}$, $V_{R_1}, V_L, V_C, V_{R_2}$ at $t=0^-$
2. " " " $t=0^+$
3. for $t \geq 0^+$

## 1. $t=0^-$



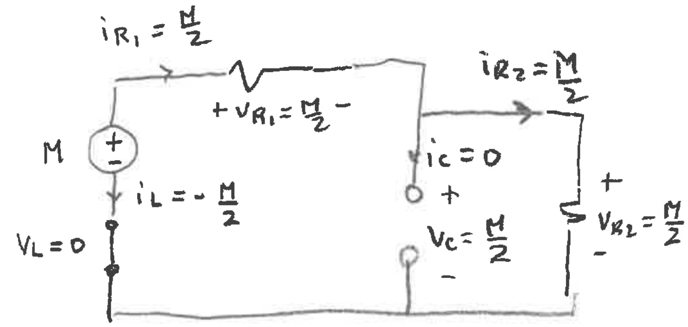

$$V_{R_1} = i_{R_1} = i_{R_2} = \frac{M}{1+1} = \frac{1}{2}M$$

$$i_L = -i_{R_1} = -\frac{1}{2}M$$

$$V_L = 0$$

$$V_C = V_{R_2} = i_{R_2} = \frac{1}{2}M$$

summary: At $t=0^-$, we have



# Example 32



## 2 $t = 0^+$

**Remember:**

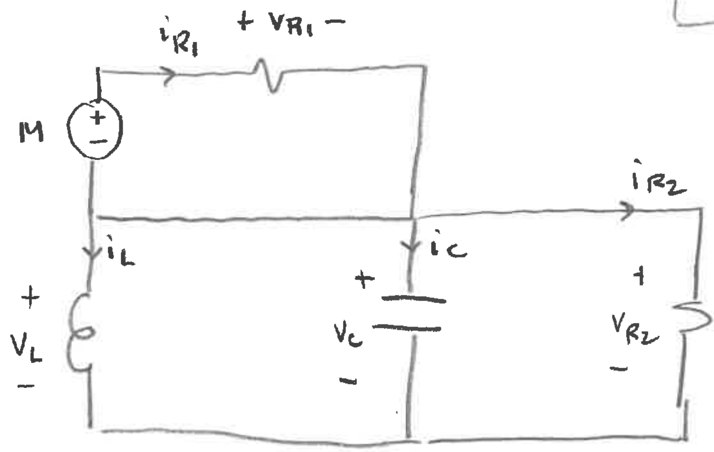
$i_C = C\frac{dv_C}{dt}$

Capacitor voltages can't jump (would result in $\infty$ $i_C$!!!)
Inductor currents " " (would result in $\infty$ $v_L$!!!)

$v_L = L\frac{di_L}{dt}$

- capacitor currents & inductor voltages can jump!
- resistor voltages & currents can jump

A




1st things 1st: Capacitor voltages & inductor currents can't jump!

$$\Rightarrow \quad i_L(0^+) = i_L(0^-) = -\tfrac{1}{2}M$$

$$v_C(0^+) = v_C(0^-) = \tfrac{1}{2}M \quad \Rightarrow \quad v_L(0^+) = v_{R_2}(0^+) = i_{R_2}(0^+) = v_C(0^+) = \tfrac{1}{2}M$$

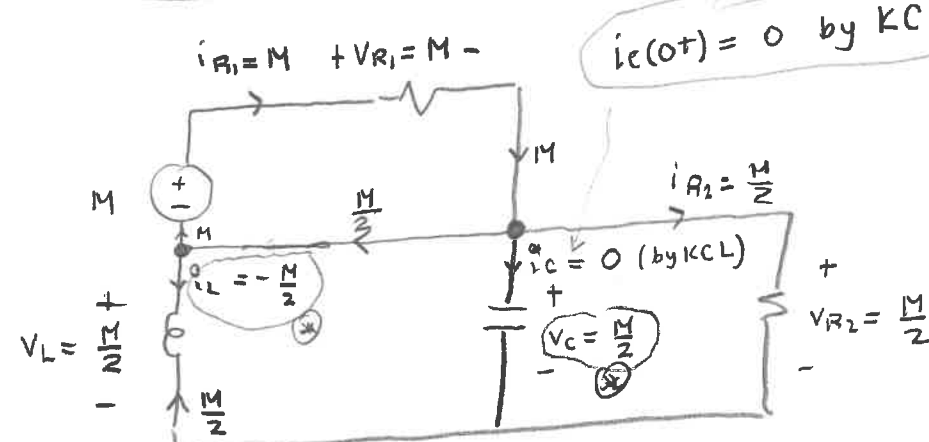
$$v_{R_1}(0^+) = i_{R_1}(0^+) = M$$

Note: $\frac{di_L}{dt}(0^+) = \frac{v_L(0^+)}{L} = \frac{v_C(0^+)}{1} = \frac{M}{2}$

$\frac{dv_C}{dt}(0^+) = \frac{i_C(0^+)}{C} = \frac{0}{1} = 0$

Summary: At $t = 0^+$, we have

$i_C(0^+) = 0$ by KCL




(*) inductor current $i_L$ & capacitor voltage $v_C$ can't jump at $t = 0$!

Example 32



3 $t \geq 0^+$

A

$v_{R_1} = i_{R_1} = M$ for $t \geq 0^+$

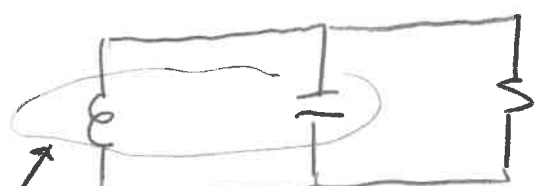
$\left\{ \begin{matrix} v_L, v_C, v_{R_2} \\ i_L, i_C, i_{R_2} \end{matrix} \right\}$ all have the form $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

where $s_1, s_2$ are roots of the characteristic eq $\Phi$.

Lets find $\Phi$ & roots $s_{1,2}$ !

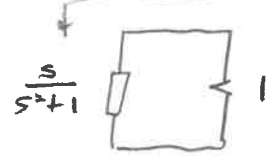
Finding $\Phi$ & Characteristic Roots

for $t \geq 0^+$ we have



$R_2 = L = C = 1$

$s \| \frac{1}{s} = \frac{(s)(\frac{1}{s})}{s + \frac{1}{s}} = \frac{1}{s + \frac{1}{s}} = \frac{s}{s^2+1}$

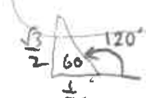



$$Z = \frac{s}{s^2+1} + 1 = \frac{s + s^2 + 1}{s^2+1} = 0$$

$$\Rightarrow \Phi_{CL} = s^2 + s + 1 = 0$$

$$\Rightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

Let $s_0 \triangleq -\frac{1}{2} + j\frac{\sqrt{3}}{2}$




$|s_0| = 1$

$\angle s_0 = 120°$

Lets now find $\left\{ \begin{matrix} i_L, i_C, i_{R_2} \\ v_L, v_C, v_{R_2} \end{matrix} \right\}$ - - - -

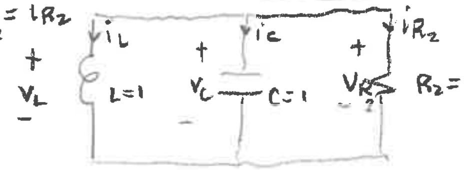
# Example 32

## Inductor

A

$V_L = s_0 i_L = v_c = V_{R_2} = i_{R_2}$ (with $L=1$, $R_2=1$)

$\angle V_L = \angle s_0 + \angle i_L = 120° + \angle i_L$




$i_C = s_0 V_C = s_0 V_L$

$\angle i_C = 120° + \angle V_L$

↖ Once we find $i_L$, the rest ~~follow very easily~~ !!!!

3040

$$i_L(0^+) = -\frac{M}{2} \qquad \frac{di_L}{dt}(0^+) = \frac{M}{2}$$

$$i_L(t) = \textcircled{A}\, e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t} + * = 2\,\mathrm{Re}\left[A\, e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}\right]$$

($A = |A|e^{j\angle A}$)

$$= |A| e^{-\frac{1}{2}t} e^{j(\frac{\sqrt{3}}{2}t + \angle A)} + *$$

$$= 2\,\mathrm{Re}\; |A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A\right)$$

$$\Rightarrow \boxed{i_L(t) = 2|A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A\right)}$$

$$i_L(0^+) = -\frac{M}{2} = 2|A| \cos\angle A \quad \Rightarrow \quad \boxed{|A|\cos\angle A = -\frac{M}{4}}$$

($|A|\cos\angle A = \mathrm{Re}\,A$)

$$i_L = A e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t} + \bar{A} e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t}$$

$$\frac{di_L}{dt} = A e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}\left[j\frac{\sqrt{3}}{2}\right] + \bar{A} e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t}\left[-j\frac{\sqrt{3}}{2}\right]$$

$$\frac{di_L}{dt}(0^+) = \frac{M}{2} = A\left[j\frac{\sqrt{3}}{2}\right] - \bar{A}\left[j\frac{\sqrt{3}}{2}\right] = j\frac{\sqrt{3}}{2}\left[A - \bar{A}\right]$$

$$= j\frac{\sqrt{3}}{2}\left[j2\,\mathrm{Im}\,A\right]$$

$$= j\frac{\sqrt{3}}{2}\left[j2\,|A|\sin\angle A\right]$$

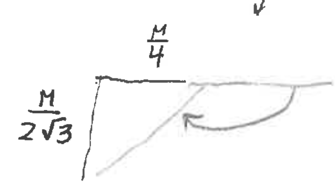
$$= -\sqrt{3}\,|A|\sin\angle A$$

$$\Rightarrow |A|\sin\angle A = -\frac{M}{2\sqrt{3}}$$

($|A|\sin\angle A = \mathrm{Im}\,A$)

$$\left\{\begin{aligned} \mathrm{Re}\,A &= -M/4 \\ \mathrm{Im}\,A &= -\frac{M}{2\sqrt{3}} \end{aligned}\right\} \Rightarrow A = -\frac{M}{4} - j\frac{M}{2\sqrt{3}}$$

drawn for $M>0$



# Example 32

3050

**A** $\Rightarrow$ $|A| = M\sqrt{\frac{1}{16} + \frac{1}{4(3)}} = M\sqrt{\frac{12+16}{(16)(12)}}$ ← 28

$$= M\sqrt{\frac{(4)(7)}{(4)(4)(4)(3)}}$$

$$= \frac{M}{4}\sqrt{\frac{7}{3}} \quad (\text{OK since } M>0)$$

$$\angle A = -180 + \text{Tan}^{-1}\left(\frac{\frac{M}{2\sqrt{3}}}{M/4}\right)$$

$$= -180' + \text{Tan}^{-1}\left(\frac{2}{\sqrt{3}}\right) \quad \text{— } 49.1066^\circ$$

$$\cong -130.8934^\circ$$

90 + 40.8934

$$i_L(t) = \frac{M}{2}\sqrt{\frac{7}{3}}\, e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t - 130.8934^\circ\right)$$

$\cos(x-90) = \sin x$

$$= \frac{M}{2}\sqrt{\frac{7}{3}}\, e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t - 40.8934^\circ\right)$$

easier to differentiate complex exponentials

$$v_L = L\frac{di_L}{dt} = \frac{di_L}{dt} = \frac{d}{dt}\left[2\,\text{Re}\; A e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}\right]$$

$$= 2\text{Re}\left\{\frac{d}{dt}\left(A\, e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}\right)\right\}$$

$$= 2\text{Re}\left\{A\, e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right\}$$

$\frac{\sqrt{3}}{2}$, 60, 120, $\frac{1}{2}$

$$= 2\text{Re}\left\{|A|e^{j\angle A}\, e^{-\frac{1}{2}t}\, e^{j\frac{\sqrt{3}}{2}t}\, 1e^{j120^\circ}\right\}$$

$$= 2\text{Re}\left\{|A|e^{-\frac{1}{2}t}\, e^{j(\frac{\sqrt{3}}{2}t + \angle A + 120^\circ)}\right\}$$

# Example 32

3060

$V_L = s_0 I_L \Rightarrow \angle V_L = \angle I_L + \angle s_0 = -130.8934 + 120 = -10.8934$

[A]

$$\Rightarrow V_L(t) = 2|A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A + 120^\circ\right)$$

($|A| = \frac{M}{4}\sqrt{\frac{7}{3}}$, $\angle A = -130.8934^\circ$)

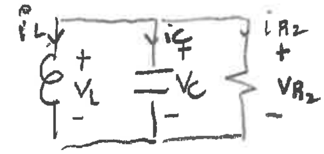
$$= \frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 10.8934^\circ\right)$$

Note that $V_L$ differs from $i_L$ by just $\angle s_0 = 120^\circ$. This is because

$V_L = s_0 L I_L$
$= s_0 I_L$

$\because s_0 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$

$|s_0| = 1$, $\angle s_0 = 120^\circ$




Note: $V_{R_2}(t) = i_{R_2}(t) = V_C(t) = V_L(t)$

$$i_C(t) = -i_L(t) - i_{R_2}(t)$$

$$= -\frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 130.8934^\circ\right) - \frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 10.8934^\circ\right)$$

$$= -\frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}t} \left\{ Re\left[\left(1 e^{-j130.8934^\circ} + 1 e^{-j10.8934^\circ}\right) e^{j\sqrt{\frac{3}{2}}t}\right]\right\}$$

MATLAB: $0.3273 - j0.9449$

MATLAB: $0.9999 e^{-j70.8934}$ ($\sim 1$)

$$= -\frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}} (1) \cos\left(\sqrt{\frac{3}{2}}t - 70.8934^\circ\right)$$

$-\cos x = \cos(x + 180)$

$$= \frac{M}{2}\sqrt{\frac{7}{3}} e^{-\frac{1}{2}} (1) \cos\left(\sqrt{\frac{3}{2}}t + 110.8934^\circ\right)$$

($180 - 70.8934^\circ$; $\angle s_0 - 10.8934^\circ$ with $\angle s_0 = 120^\circ$)

Note:

$i_C = s_0 C V_C$
$= s_0 V_L \Rightarrow$ It follows that $i_C$ will differ from $V_L$ just by $\angle s_0 = 120^\circ$

$\angle i_C = \angle s_0 + \angle V_L$
$= 120^\circ - 10.8934$
$= 110.8934^\circ$

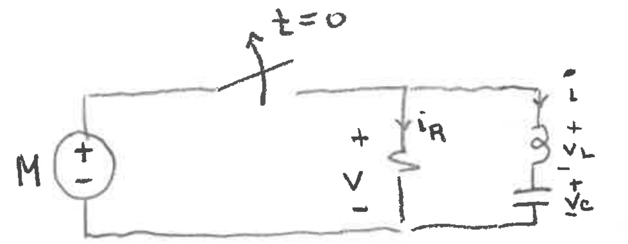
3060

# Example 32



B

Assume $M > 0$



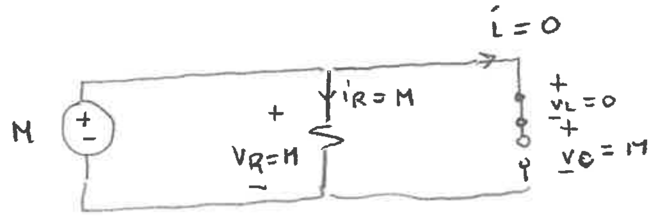

(All R = L = C = 1)

Find $v, i_R, i, v_L, v_c$ ① at $t = 0^-$

② at $t = 0^+$

③ for $t \geq 0^+$

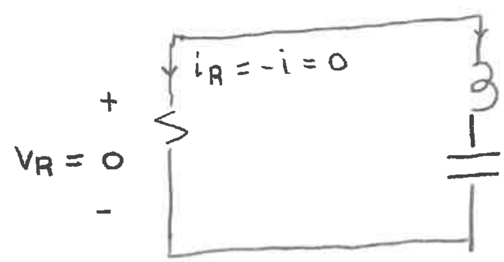
① $t = 0^-$




② $t = 0^+$

since inductor current can't jump!

$i(0^+) = i(0^-) = 0$ ($i_L$)




$v_L = V_R - V_c = 0 - M = -M$

$v_c(0^+) = V_c(0^-) = M$

since capacitor voltage can't jump!

(All R = L = C = 1)

$$\frac{dV_c(0^+)}{dt} = \frac{i(0^+)}{c} = 0$$

$$\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{-M}{1} = -M$$

③ $t \geq 0^+$

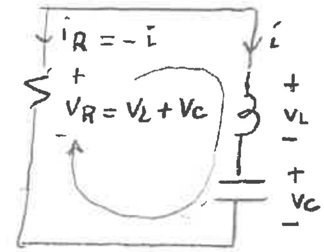
$v, i_R, i, v_L, v_c$ all have the form $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

where $s_{1,2}$ are roots of the characteristic eq $\Phi$.

Lets find $\Phi$ & the roots $s_{1,2}$!

Example 32

3080



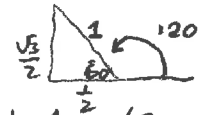

$(R = L = C = 1)$

## Finding $\Phi$ & Characteristic Roots

B

$$Z = 1 + s + \frac{1}{s} = \frac{s^2 + s + 1}{s}$$

$$\Rightarrow \Phi = \text{numerator of } Z = s^2 + s + 1 = 0 \Rightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$




$$\Rightarrow s_0 \triangleq -\frac{1}{2} + j\frac{\sqrt{3}}{2} \qquad |s_0| = 1 \qquad \angle s_0 = 120°$$

**Note:**

Suppose we find $V_C$. $\Rightarrow$ Since $i = s_0 V_C \Rightarrow \angle i = \angle s_0 + \angle V_C = 120° + \angle V_C$

$V_L = s_0 \dot{i} \Rightarrow \angle V_L = \angle s_0 + \angle i = 120° + \angle i$

Multiplying by $s_0$ is simpler than differentiation & greatly simplifies our arithmetic/algebraic steps !!!!

$\Rightarrow$ By finding $V_C$, it will be easy to determine $i$, $i_R = -i$, & $V_L$ from $V_C$!

Lets find $V_C$.

**Finding $V_C$ from initial conditions**

$$V_C(t) = A e^{s_0 t} + * = 2\,\mathrm{Re}\left[A e^{s_0 t}\right]$$

($A = |A| e^{j\angle A}$, $s_0 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$)

$$= 2\,\mathrm{Re}\left[|A| e^{-\frac{1}{2}t} e^{j(\frac{\sqrt{3}}{2}t + \angle A)}\right]$$

$$= 2|A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A\right)$$

To find $A$ ($|A|$ & $\angle A$) we use

$$V_C(0^+) = M$$
$$\dot{V}_C(0^+) = 0$$

} found earlier (previous page)

Noting that $\dot{V}_C = s_0 A e^{s_0 t} + * = 2\,\mathrm{Re}\left[s_0 A e^{s_0 t}\right]$

$$= 2\,\mathrm{Re}\left[|s_0| e^{j\angle s_0} |A| e^{j\angle A} e^{(-\frac{1}{2} + j\frac{\sqrt{3}}{2})t}\right]$$

$$= 2\,\mathrm{Re}\left[1\,|A| e^{-\frac{1}{2}t} e^{j(\frac{\sqrt{3}}{2}t + \angle A + \angle s_0)}\right]$$

$$= 2|A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A + 120°\right)$$

From the above we have

$$V_C(0^+) = M = 2|A| \cos(\angle A) \Rightarrow |A|\cos\angle A = \frac{M}{2} \quad (= \mathrm{Re}\,A)$$

$$\dot{V}_C(0^+) = 0 = 2|A| \cos(\angle A + 120°) \Rightarrow \cos(\angle A + 120°) = 0$$

Example 32



**B**

$v_L(0+) = -M = 2|C|\cos(\angle C)$ ← This tells us that $\angle C$ must be a 2nd or 3rd Quadrant angle because $\cos\angle C < 0 \Leftrightarrow \angle C$ is in 2nd or 3rd Quad

($\angle C = \angle v_L$)

$$\dot{v}_L = s_0 C e^{s_0 t} + * = 2\,\mathrm{Re}\left[s_0 C e^{s_0 t}\right]$$

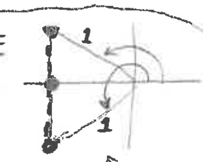
$$= 2\,\mathrm{Re}\left[|s_0||C|e^{-\frac{1}{2}t} e^{j\left(\frac{\sqrt{3}}{2}t + \angle C + \angle s_0\right)}\right]$$

($|s_0| = 1$, $\angle s_0 = 120$)

$$= 2|C|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle C + 120^\circ\right)$$

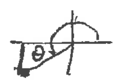
($\angle v_L + 120^\circ$)

Visualization:



↑ For either of these $\cos\angle C < 0$

Here, $\cos\angle C = \cos(180 - \theta) = -\cos\theta < 0$



Here, $\cos\angle C = \cos(180^\circ + \theta) = -\cos\theta < 0$

$$\dot{v}_L(0+) = M = 2|C|\cos(\angle C + 120^\circ)$$

$$= 2|C|\left[\cos\angle C\cos 120^\circ - \sin\angle C\sin 120^\circ\right]$$

$$= 2|C|\left[\cos\angle C\left(-\tfrac{1}{2}\right) - \sin\angle C\left(\tfrac{\sqrt{3}}{2}\right)\right]$$

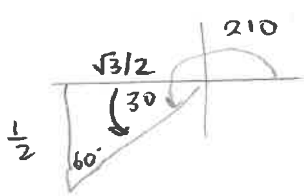
$$M = -2|C|\cos\angle C = 2|C|\left[-\tfrac{1}{2}\cos\angle C - \tfrac{\sqrt{3}}{2}\sin\angle C\right]$$

$\Rightarrow \quad 2\cos\angle C = \cos\angle C + \sqrt{3}\sin\angle C$

$\Rightarrow \quad \cos\angle C = \sqrt{3}\sin\angle C$

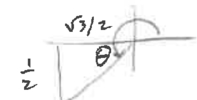
$\Rightarrow \quad \tan\angle C = \frac{1}{\sqrt{3}} > 0 \Rightarrow \tan > 0$ in 3rd Quad

$\Rightarrow \angle C$ is a 3rd Quad angle



$\Rightarrow \angle C = 210^\circ$

Why? $\frac{1}{\sqrt{3}} = \tan\angle C = \tan(180 + \theta) = \tan\theta \Rightarrow \theta = 30^\circ \Rightarrow \angle C = 180 + 30 = 210^\circ$



$$\Rightarrow -M = 2|C|\cos\angle C = 2|C|\cos 210^\circ = 2|C|\left(-\tfrac{\sqrt{3}}{2}\right) = -\sqrt{3}|C| \Rightarrow |C| = \frac{M}{\sqrt{3}}$$

$$\Rightarrow C = \frac{M}{\sqrt{3}} e^{j210^\circ}$$

(180+30)

$v_L = 2|C|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle C\right) \Rightarrow$

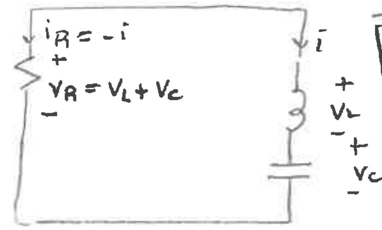
$$v_L = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + 210^\circ\right) = -\frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + 30^\circ\right)$$

☺ agrees with what we found earlier above!

## Example 32

3100

B




$(R = L = C = 1)$

### Finding $i$ (from $v_C$)

$$i = i_L = i_C = C\frac{dv_C}{dt} = \dot{v}_C \quad \overset{\text{or}}{\Longrightarrow} \quad i = s_0 V_C$$

$$\Longrightarrow \quad |i| = |s_0|\,|V_C| = |V_C|$$

$$\angle i_C = \angle s_0 + \angle V_C$$

$$= 120 + \angle V_C$$

$$v_C(t) = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 30^\circ\right) \Longrightarrow$$

$$i(t) = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t \underbrace{- 30^\circ + 120^\circ}_{+90^\circ}\right)$$

$$= -\frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

### Finding $i$ (from $i_L$ initial conditions)

$$i_L = i = Be^{s_0 t} + * = 2\,\mathrm{Re}\left[\underset{|B|e^{j\angle B}}{B}\, e^{\overset{-\frac{1}{2}+j\frac{\sqrt{3}}{2}}{s_0} t}\right] = 2\,\mathrm{Re}\left[|B| e^{-\frac{1}{2}t} e^{j\left(\frac{\sqrt{3}}{2}t + \angle B\right)}\right]$$

$$= 2|B| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \underset{\angle i}{\angle B}\right)$$

Now we can find $B$ ($|B|$ & $\angle B$) from the initial conditions:

$$\dot{i}_L(0^+) = 0$$
$$\frac{di_L}{dt}(0^+) = -M$$

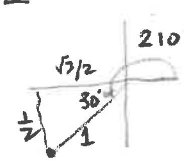
(found earlier!)

$$\frac{di_L}{dt} = s_0 B e^{s_0 t} + * = 2\,\mathrm{Re}\left[s_0 B e^{s_0 t}\right] = 2\,\mathrm{Re}\left[\underset{1}{|s_0|}\,|B|\, e^{-\frac{1}{2}t} e^{j\left(\frac{\sqrt{3}}{2}t + \angle B + \overset{120^\circ}{\angle s_0}\right)}\right]$$

$$= 2|B| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \underset{\angle i + 120^\circ}{\angle B + 120^\circ}\right)$$

$$\Longrightarrow i_L(0^+) = 0 = 2|B|\cos\angle B \Longrightarrow \mathrm{Re}\,B = |B|\cos\angle B = 0 \Longrightarrow \angle B = \pm 90^\circ$$

$$\frac{di_L}{dt}(0^+) = -M = 2|B|\cos(\angle B + 120^\circ) \Longrightarrow |B| = \frac{-M}{2\cos(\angle B + 120^\circ)} \overset{\angle B = 90}{=} \frac{-M}{2\cos 210^\circ}$$




$$= \frac{-M}{2\left(-\frac{\sqrt{3}}{2}\right)} = \frac{M}{\sqrt{3}} > 0 \quad ☺$$

With $\angle B = -90^\circ$, we get $|B| = \frac{-M}{2\cos(30)} = \frac{-M}{2\left(\frac{\sqrt{3}}{2}\right)} = -\frac{M}{\sqrt{3}} < 0 \Rightarrow$ can't be since $M > 0$ & $|B|$ must be $\geq 0$!

# Example 32



**B**

$\mathrm{Re}\,A = |A| \cos \angle A = \frac{M}{2}$

$\cos(\angle A + 120^\circ) = 0$

↓

lets determine $\angle A$ & $|A|$ 2 ways . . .

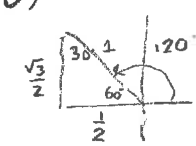
**i** Trig Formula

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(\angle A + 120^\circ) = \cos \angle A \cos(120) - \sin \angle A \sin(120^\circ) = 0$

$\Rightarrow \sin \angle A \sin(120^\circ) = \cos \angle A \cos(120^\circ)$

$\Rightarrow \tan \angle A \tan 120^\circ = 1$




$\tan 120 = \tan(180 - 60^\circ)$
$= -\tan 60^\circ$
$= -\sqrt{3}$

$\Rightarrow \tan \angle A = \frac{1}{\tan 120} = \frac{-1}{\sqrt{3}}$

$\Rightarrow \angle A = -30^\circ$

$|A| \cos \angle A = \frac{M}{2} \quad \Rightarrow \quad |A| = \frac{M}{2 \cos \angle A} = \frac{M}{2(\cos(-30^\circ))} = \frac{M}{\sqrt{3}}$ (with $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$)

$\Rightarrow A = \frac{M}{\sqrt{3}} e^{-j30^\circ} \quad \left(|A| = \frac{M}{\sqrt{3}} \quad \angle A = -30^\circ\right)$

$\Rightarrow V_c = 2|A| e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + \angle A\right) \Rightarrow \boxed{V_c = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 30^\circ\right)}$

**ii** Cosine Angle

$\cos(\angle A + 120^\circ) = 0 \Rightarrow \angle A + 120^\circ = \pm 90^\circ \Rightarrow \angle A = -30^\circ, -210^\circ$

or $= \pm 270 \Rightarrow \angle A = 150^\circ$ , $-390^\circ$

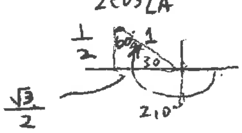
$150^\circ = -210 + 360^\circ$; $-390^\circ = -30^\circ - 360$

$\angle A = -30^\circ$:

$|A| = \frac{M}{2\cos\angle A} = \frac{M}{2\,\frac{\sqrt{3}}{2}} = \frac{M}{2}$ (ok since $M > 0$)

$\angle A = -210$:

$|A| = \frac{M}{2\cos\angle A} = \frac{M}{2\left(-\frac{\sqrt{3}}{2}\right)} = \frac{-M}{\sqrt{3}}$ (NOT OK unless $M < 0$)




$\Rightarrow$ it must be that $\angle A = -30^\circ$

$\Rightarrow A = \frac{M}{\sqrt{3}} e^{-j30^\circ}$ (as we got above)

# Example 32



**B** with $\angle B = +90^\circ$, we now have $|B| = \frac{M}{\sqrt{3}}$ $

$i_L = i = 2|B|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle B\right)$

$$i_L(t) = i(t) = \frac{2M}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}t + 90^\circ\right)$$
$$= -\frac{2M}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

which is precisely what we got earlier above!

## Find $v_L$ from $i_L$

$v_L = L\frac{di_L}{dt} \Rightarrow V_L = s_0 L I_L = s_0 I_L \Rightarrow |V_L| = |s_0||I_L| = |I_L|$ (with $|s_0| = 1$)

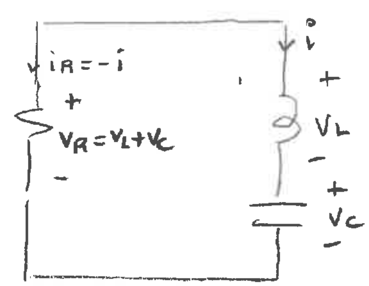
$\angle V_L = \angle s_0 + \angle I_L = 120^\circ + \angle I_L$

$$\Rightarrow v_L(t) = \frac{2M}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}t + (90^\circ + 120^\circ)\right)$$

($90^\circ + 120^\circ = 210^\circ = 180^\circ + 30^\circ$; $\cos(180 + x) = -\cos x$)

$$= -\frac{2M}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}t + 30^\circ\right)$$

## Find $v_L$ from Initial Conditions




We know $v_L(0^+) = -M$ $ $i_L(0^+) = 0$

$v_L = v_R - v_C$

$\dot{v}_L = \dot{v}_R - \dot{v}_C$

$= -\frac{di_L}{dt} - i_L$

$\Rightarrow \dot{v}_L = -v_L - i_L \Rightarrow \dot{v}_L(0^+) = -v_L(0^+) - i_L(0^+)$

$= M - 0$

$= M$

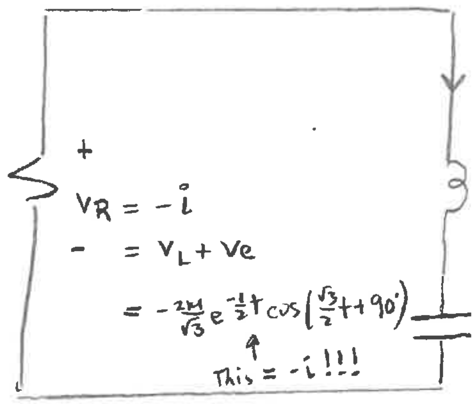
$\Rightarrow$ We'll now use $\boxed{v_L(0^+) = -M, \ \dot{v}_L(0^+) = M}$ to determine $c = |c|e^{j\angle c}$ in

$v_L = ce^{s_0 t} + * = 2\,\mathrm{Re}[Ce^{s_0 t}]$

$= 2|c|e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t + \angle c\right)$

B

## Visual Summary




$V_R = -i$

$= V_L + V_C$

$= -\frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + 90^\circ\right)$

↑ This $= -i$ !!!

$i = i_L = i_C = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 30^\circ + 120^\circ\right)$ — $90^\circ$; $\angle V_C + \angle s_0 = -30^\circ + 120^\circ$

$V_L = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t + 90 + 120\right)$ — $210^\circ$; $\angle i + \angle s_0 = 90 + 120^\circ$

$V_C = \frac{2M}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - 30^\circ\right)$

↑ $\angle i - \angle s_0$

$= 90 - 120^\circ$

$= -30^\circ$

Now lets show that

$V_R = -i = V_L + V_C$

$$-\mathrm{Re}\left[\frac{2M}{\sqrt{3}} e^{s_0 t} e^{j(-30 + \angle s_0)}\right] = \mathrm{Re}\left[\frac{2M}{\sqrt{3}} e^{s_0 t} e^{j(90 + \angle s_0)} + \frac{2M}{\sqrt{3}} e^{s_0 t} e^{-j30^\circ}\right]$$

$$-e^{j(-30^\circ + 120^\circ)} \stackrel{?}{=} e^{j(90 + 120^\circ)} + e^{-j30^\circ}$$

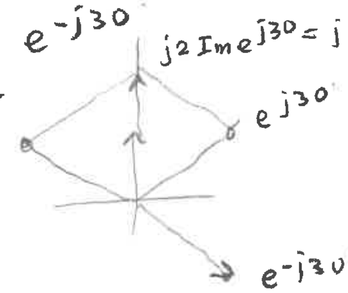
$$-e^{-j30^\circ} e^{j90} e^{j30} = e^{j90} e^{j90} e^{j30} + e^{-j30}$$

$$-j = -e^{j30} + e^{-j30}$$

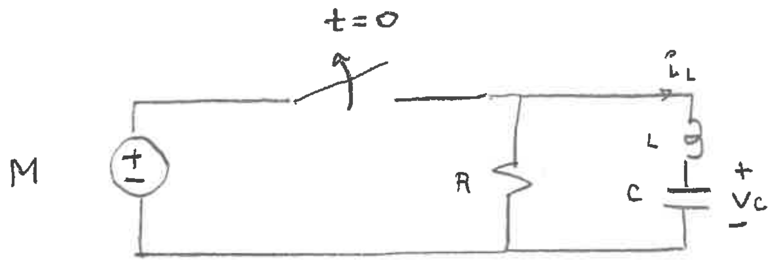
$$j = e^{j30} - e^{-j30} = j2 \sin 30 = j1 \checkmark\checkmark\checkmark$$

($\sin 30 = 1/2$)

$j2\,\mathrm{Im}\, e^{j30} = j2 \sin 30 = j1$ ($\sin 30 = \frac{1}{2}$)



Problem 32 (Switching Circuits) 3140




$R=2$, $L=1$, $C=\frac{1}{2}$

Find $i_L$, $V_C$

① at $t=0^-$

② at $t=0^+$

③ for $t \geq 0^+$