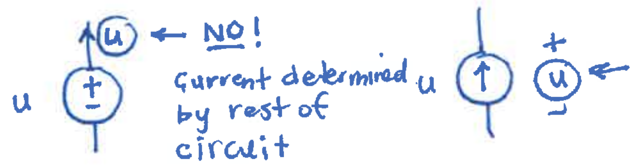
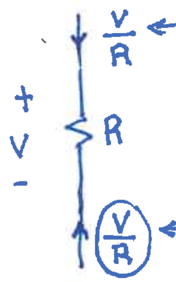
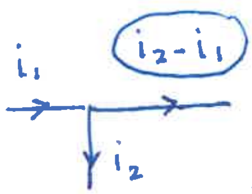
(Will) Errors that Can NOT be Tolerated!

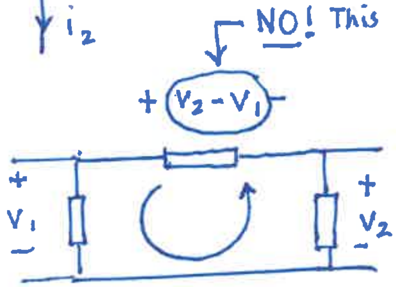
Be Forewarned: These type of errors generally lead to a final grade of D or F!!!

PLEASE Don't Ever Make Any of These Errors!!!... (PLEASE!!!)









NO!

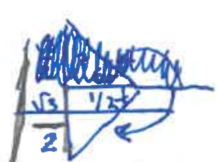
$\angle(-1+j) \neq \tan^{-1}\left(\frac{1}{-1}\right) = -45°$!

$\angle(-1+j) =$  $= 180° - \tan^{-1}\left(\frac{1}{1}\right) = 180 - 45° = 135°$

Yes!

NO!

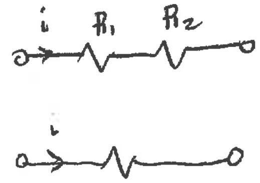
$\angle\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \neq \tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = \tan^{-1}(\sqrt{3}) = 60°$

$\angle\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) =$  $= -180° + \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = -180 + 60° = -120°$

Yes!

$\frac{u}{s+\frac{1}{s}} \neq \frac{u}{s} + \frac{u}{\frac{1}{s}}$ (You MUST be able to do basic algebra!)

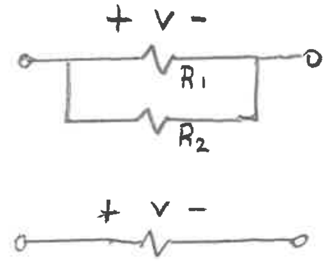
Please don't ever confuse the following:



$R_{series} = R_1 + R_2$

(same current i passes through $R_1, R_2, \text{ \& } R_{series}$!)

and

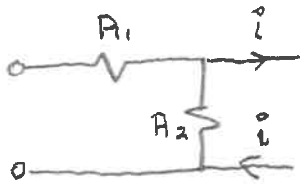


$R_{parallel} = R_1 \| R_2 \triangleq \dfrac{R_1 R_2}{R_1 + R_2}$

or

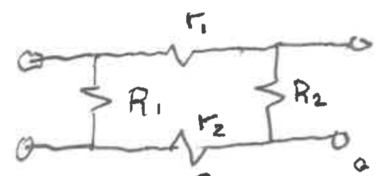
$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2}$$

(same voltage v is across $R_1, R_2, R_{parallel}$!)



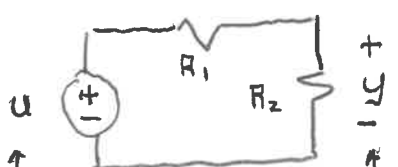
Here, R_1 & R_2 are in series iff $i = 0$!!!

(so that same current passes through R_1 & R_2!)



Here, R_1 & R_2 are in parallel iff $r_1 = r_2 = 0$

(so that same voltage appears across R_1 & R_2!)

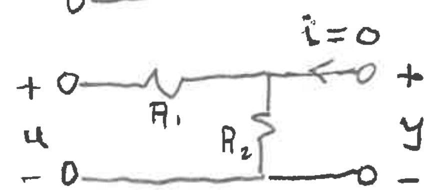


input voltage

output voltage

is equivalent to

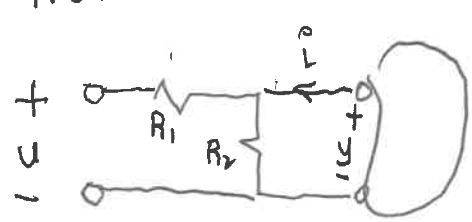
" " "



input voltage

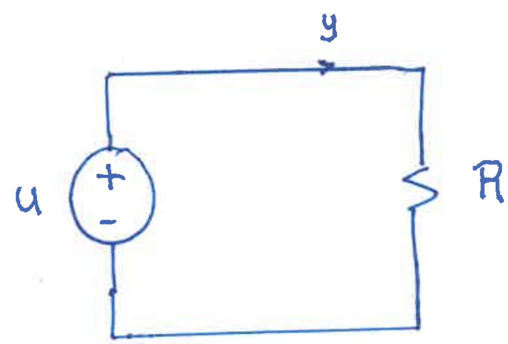
output voltage

The above are NOT the same as:



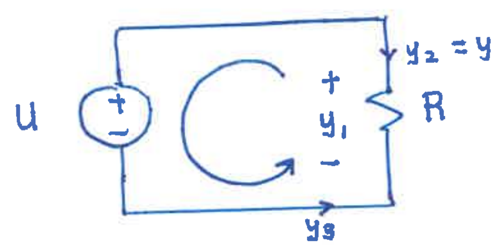
The are the same iff $i = 0$!

Example 1 (1 Resistor!)



Relate y to u ($\&$ R)

solution:



Note: 1^{st} we find y_1

then $y_2 = y$!

We can then also find $y_3 = -y$

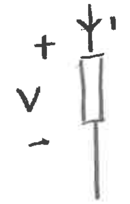
$y_1 = u$ by $\overset{\text{CCW}}{\text{KVL}}$

$$y_2 = \overset{\text{ohm}}{y} = \frac{y_1}{R} = \frac{u}{R}$$

$$\overset{\text{KCL}}{y_3} = -y_2 = -\frac{u}{R} = -y$$

Note: The numbering of the variables y_1, y_2, y_3 has been selected to indicate the ordering of the logical steps to be taken toward solving for y!

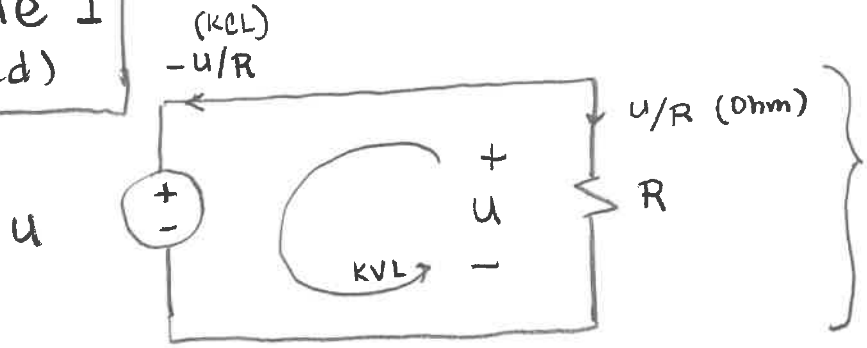
Power Concepts



$$p_{abs} \triangleq v\,i$$

$$p_{del} = -p_{abs}$$

Example 1 (continued)



This summarizes the analysis on the previous page.

$$p^{u}_{abs} = (u)\left(-\frac{u}{R}\right) = -\frac{u^2}{R} < 0$$

Negative p^{u}_{abs} means that u is actually delivering power

i.e. $p^{u}_{del} = -p^{u}_{abs} = \frac{u^2}{R} > 0$

$$p^{R}_{abs} = (u)\left(\frac{u}{R}\right) = \frac{u^2}{R} > 0$$

Note:

$$\boxed{p^{u}_{del} = p^{R}_{abs}}$$

Power deliverd by u is equal to power absorbed by R

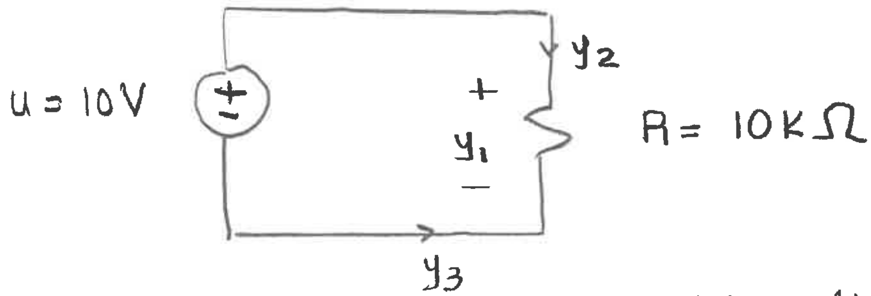
Sum of absorbed powers within a cct is always zero!

$$\boxed{p^{u}_{abs} + p^{R}_{abs} = 0}$$
$$\left(-\frac{u^2}{R}\right) \quad \left(\frac{u^2}{R}\right)$$

Sum of delivered powers within a cct is always zero!

$$\boxed{p^{u}_{del} + p^{R}_{del} = 0}$$
$$\left(\frac{u^2}{R}\right) \quad \left(\frac{-u^2}{R}\right)$$

Problem 1



Note: Always specify your associated logic or reasoning!

a) Determine y_1

b) Determine y_2

c) Determine y_3

Note: In this problem, y_2 represents "conventional" current flow! How do the electrons flow in this problem?

d) Determine power absorbed p_R^{abs} by R.

e) Determine power delivered p_R^{del} by R.

f) Determine power absorbed p_u^{abs} by u.

g) Determine power delivered p_u^{del} by u.

In this course, we focus almost exclusively on conventional current flow & not on electron current flow...

Some Basic Power Concepts

For 

$$p_{abs} \triangleq v\, i$$

$$p_{del} \triangleq -p_{abs}$$

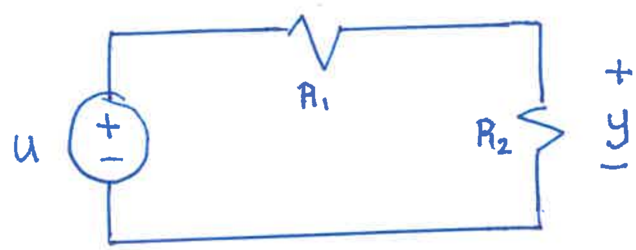
$\triangleq$ means "equals by definition"

Note:

Conventional current flow is opposite electron current flow!

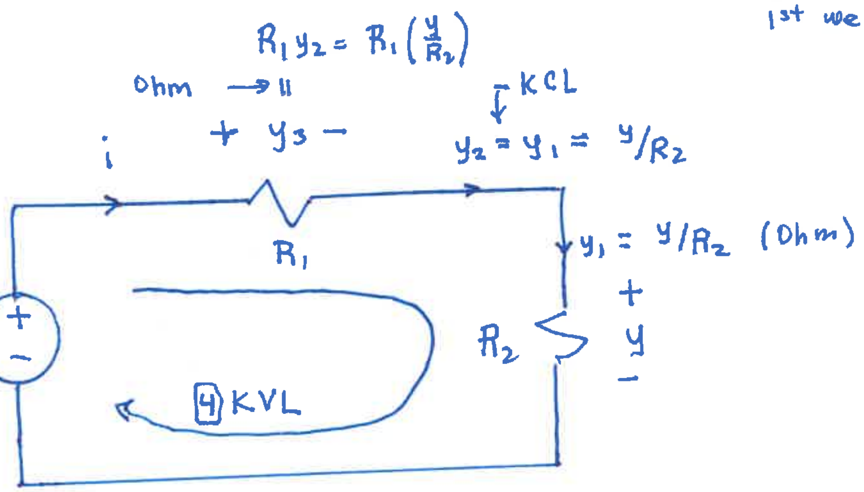
Example 2 (Voltage Divider)

Relate y to u (& R_1, R_2)



Note: 1st we find y_1 then y_2 then y_3 then step $\boxed{4}$ is a KVL that yields the desired equation relating y to R_1, R_2, u!

Solution:



NOTE: Don't ever actually write/create the unnecessary variables y_1, y_2, y_3!

PLEASE! Doing so causes GREAT Harm!

$\boxed{4}$ KVL:

$$u = \underbrace{y_3}_{\text{ohm}} + y$$

$$= R_1\left(\frac{y}{R_2}\right) + y$$

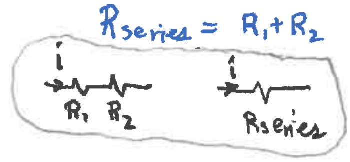
$$= \left(\frac{R_1}{R_2} + 1\right) y$$

$$= \left(\frac{R_1 + R_2}{R_2}\right) y$$

(algebra) $\Longrightarrow$ $\boxed{y = \left(\frac{R_2}{R_1 + R_2}\right) u}$ $\Longrightarrow$ $i = y_2 = y_1 = y/R_2$

$\Rightarrow$ $\boxed{i = \frac{u}{R_1 + R_2} = \frac{u}{R_{series}}}$

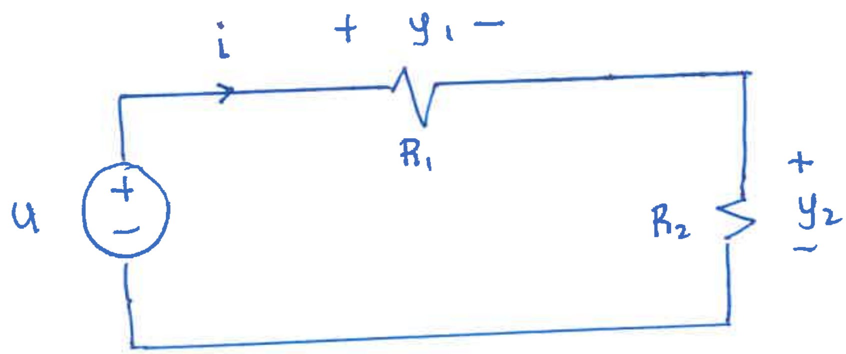
$R_{series} = R_1 + R_2$



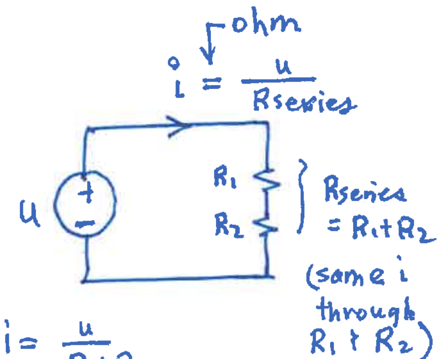
$\Rightarrow$ $y_3 = R_1 i$ $\Longrightarrow$ $\boxed{y_3 = \left(\frac{R_1}{R_1 + R_2}\right) u}$

Example 2 (Voltage Divider Summary)

Relate y_1, y_2, i to u (R_1 & R_2)



$$\boxed{i = \frac{u}{R_{series}} = \frac{u}{R_1 + R_2}}$$

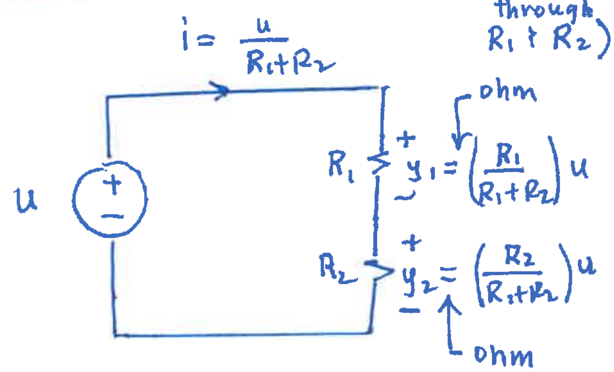


ohm

$y_1 = R_1 i \implies \boxed{y_1 = \left(\frac{R_1}{R_1+R_2}\right) u}$

ohm

$y_2 = R_2 i \implies \boxed{y_2 = \left(\frac{R_2}{R_1+R_2}\right) u}$



Problem 2

Find power absorbed & delivered by each component of circuit:

P_1 abs P_1 del ← These are for R_1

P_2 abs P_2 del ← These are for R_2

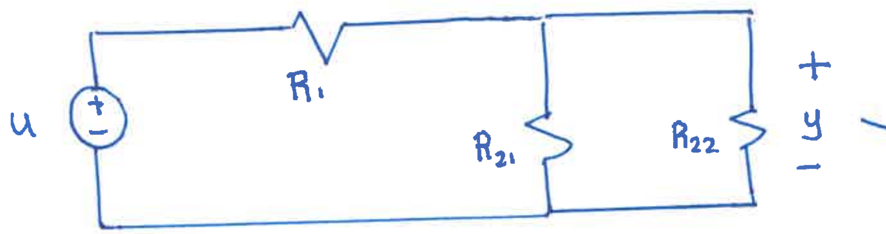
P_u abs P_u del ← These are for u

what do these add up to? $\sum P_{abs} = ?$

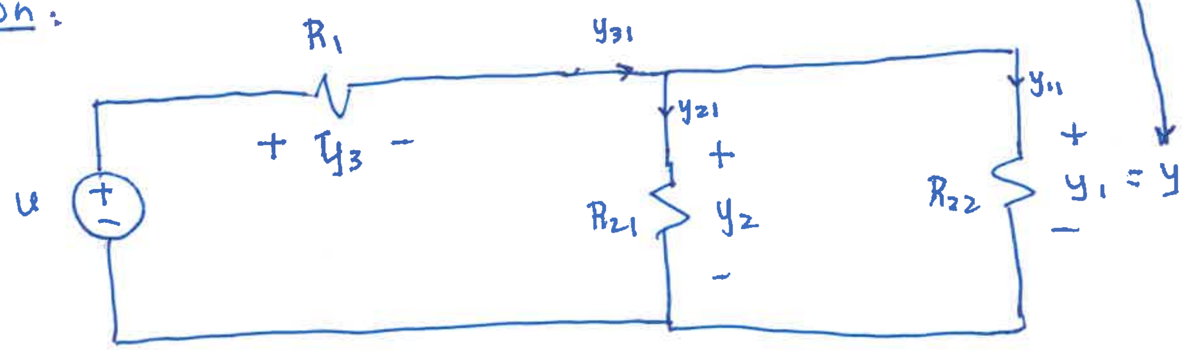
what do these add up to? $\sum P_{del} = ?$

Example 3 (Voltage Divider + 1 Extra Resistor)

Relate y to u
(↪ R_1, R_{21}, R_{22})

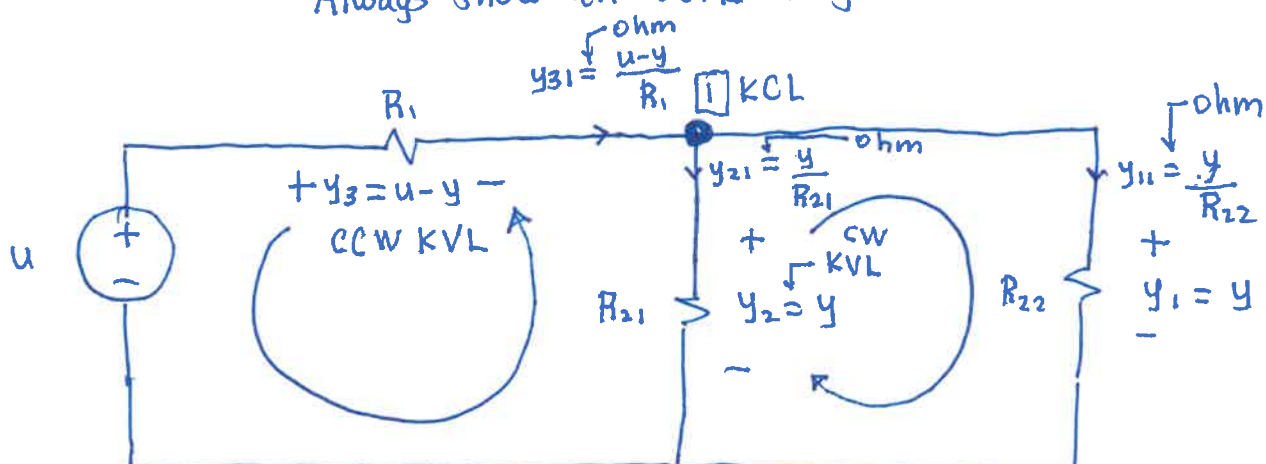


solution:

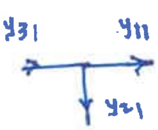


Note: the variable numbering $y_1, y_{11}, y_2, y_{21}, y_3, y_{31}$ has been selected to indicate the logical steps toward finding y!

Always show all work on your circuit:



1 KCL: (+ohm)

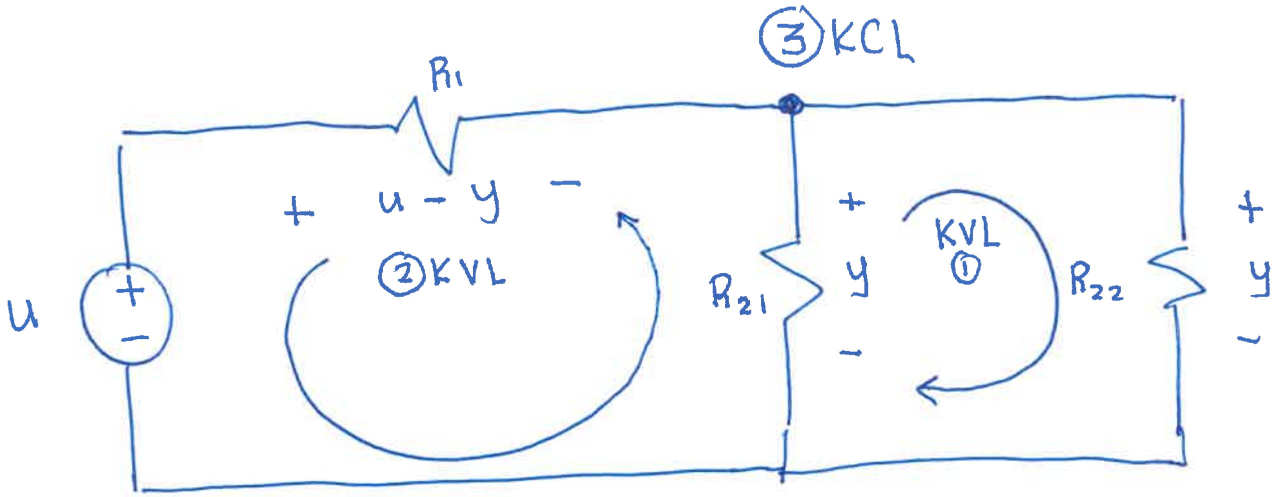


$$\boxed{\underbrace{\frac{u-y}{R_1}}_{\text{ohm}} \overset{\text{KCL}}{=} \underbrace{\frac{y}{R_{21}}}_{\text{ohm}} + \underbrace{\frac{y}{R_{22}}}_{\text{ohm}}}$$

← Desired equation relating y to u (↪ R_1, R_{21}, R_{22})

Example 3

Here is how the above should (all) look like in your homework:



③ KCL: (+ohm)

$$\boxed{\overset{\text{ohm}}{\left(\frac{u-y}{R_1}\right)} \overset{\text{KCL}}{=} \overset{\text{ohm}}{\left(\frac{y}{R_{21}}\right)} + \overset{\text{ohm}}{\left(\frac{y}{R_{22}}\right)}}$$

always place a box around eq relating y to u!

algebra:

$$\Longrightarrow \quad \frac{u}{R_1} = y\left[\frac{1}{R_1} + \frac{1}{R_{21}} + \frac{1}{R_{22}}\right]$$

$$= \frac{y}{R_1} + y\left[\frac{1}{R_{21}} + \frac{1}{R_{22}}\right]$$

R_{21} & R_{22} are said to be in parallel since the same voltage y appears across each!

$$\frac{1}{R_{21}} + \frac{1}{R_{22}} = \frac{1}{R_2}$$

$R_2 = R_{21} \| R_{22}$

$$\Rightarrow \frac{1}{R_2} = \frac{1}{R_{21}} + \frac{1}{R_{22}} = \frac{R_{21}+R_{22}}{R_{21}R_{22}}$$

$$\boxed{R_2 = R_{21} \| R_{22} = \frac{R_{21}R_{22}}{R_{21}+R_{22}}}$$

Example 3

$$\frac{1}{R_2} = \frac{1}{R_{21}} + \frac{1}{R_{22}}$$

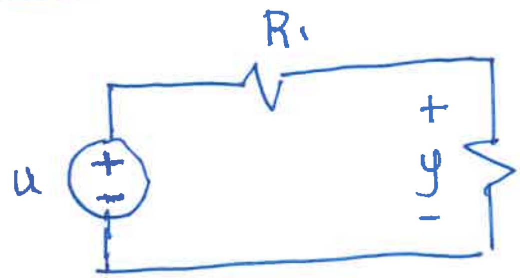
more algebra $\Longrightarrow$

$$\frac{u}{R_1} = \frac{y}{R_1} + \frac{y}{R_2}$$

where $R_2 = R_{21} \| R_{22} = \dfrac{R_{21} R_{22}}{R_{21} + R_{22}}$

Now we are back to Example 2 with

VERY CRITICAL OBSERVATION!



$R_2 \triangleq (R_{21} \| R_{22})$

solving for y $\Longrightarrow$

$$\frac{u}{R_1} = y \left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$u = y \left[1 + \frac{R_1}{R_2}\right]$$

$$= y \left[\frac{R_1 + R_2}{R_2}\right]$$

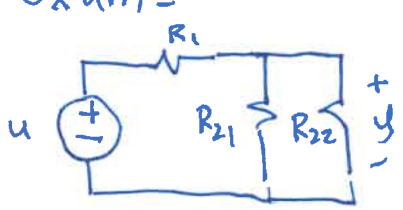
$$\Longrightarrow \boxed{y = \left[\frac{R_2}{R_1 + R_2}\right] u}$$

PLEASE LEARN THIS! (ASAP)

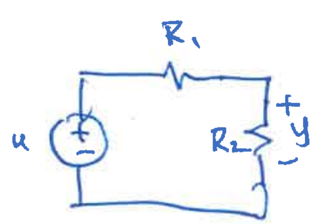
$R_2 \triangleq R_{21} \| R_{22} \triangleq \dfrac{R_{21} R_{22}}{R_{21} + R_{22}}$

Summary:

Here is how all of the above should look like on an exam:



$\Longrightarrow$



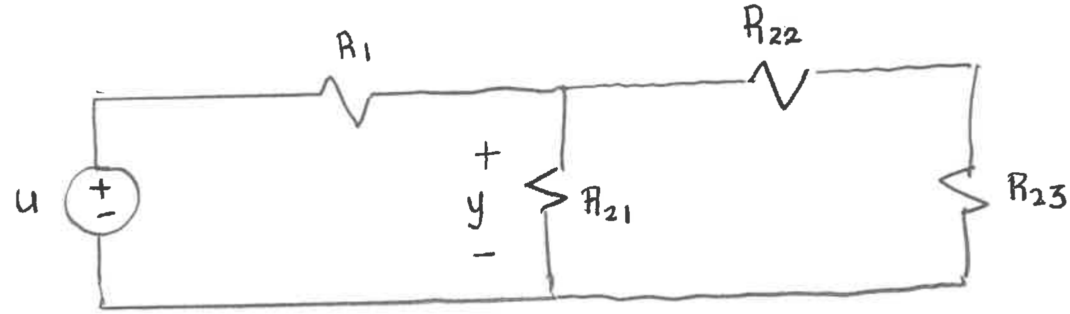
$y = \left(\dfrac{R_2}{R_1 + R_2}\right) u$ $\quad R_2 \triangleq (R_{21} \| R_{22}) \triangleq \dfrac{R_{21} R_{22}}{R_{21} + R_{22}}$

This was MUCH Faster!

Cool ???

Problem 3

Relate y to u as specified below:



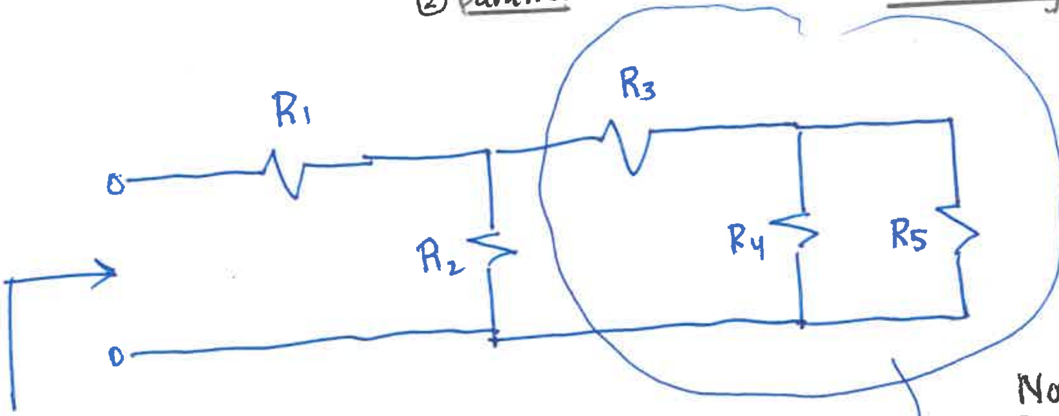
a) Relate y to u by using series-parallel concepts to make above circuit look like a voltage divider.

b) Relate y to u by using KVL, Ohm, KCL to propagate y rightward.

(You are NOT permitted to introduce any other new variables! You MUST use y & u (& the R's))

Example 4 (Finding the Equivalent Resitance)

Recall: 2 devices are in ① series iff they have the same current through them
② parallel iff " " " " voltage across "



$R_{eq} = ?$

Note: because they have same voltage across them!

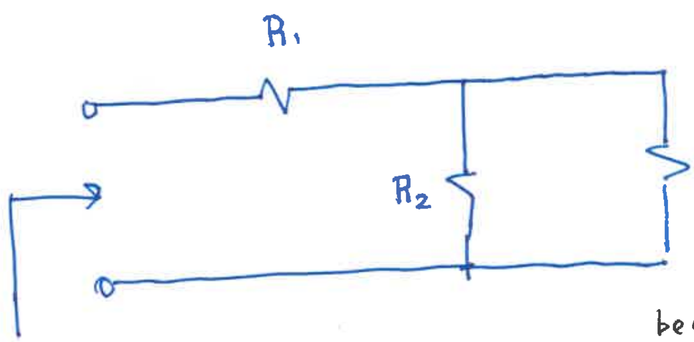
1) R_4 & R_5 are in parallel

because they have the same current through them!

2) R_3 is in series with $R_4 || R_5$

3) R_2 is in parallel with $R = [R_3 + (R_4 || R_5)]$

solution:



$R = [R_3 + (R_4 || R_5)]$

$R_{eq} = R_1 + R_2 || R$

because they have the same voltage across them!

4) R_1 is in series with $R_2 || R$

because they have the same current through them!

$$\boxed{R_{eq} = R_1 + R_2 || [R_3 + (R_4 || R_5)]}$$

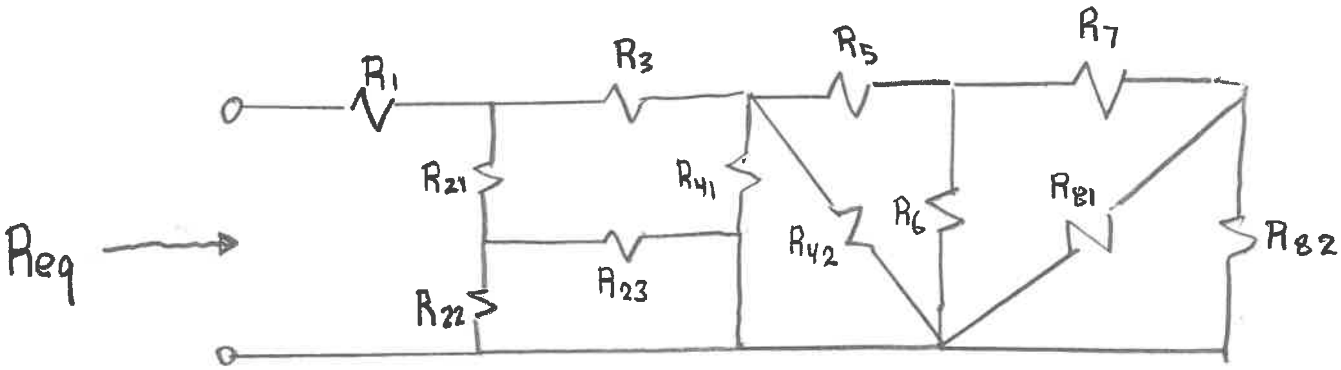
where $\boxed{A||B = \frac{AB}{A+B}}$

or $\frac{A+B}{AB} = \frac{1}{A||B}$

or $\frac{1}{A} + \frac{1}{B} = \frac{1}{A||B}$

Problem 4

Find Req



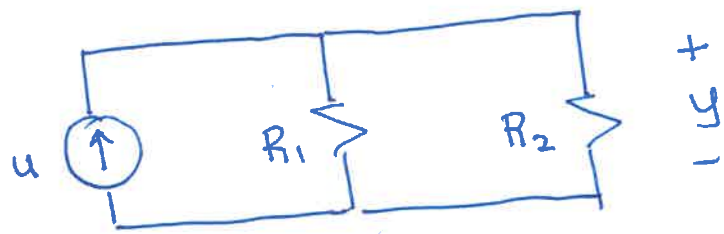
Hints: R_{81} & R_{82} are in parallel
R_{41} & R_{42} " " "
R_{22} & R_{23} " " "

Hint: Redraw circuit several times so that you can clearly show how resistors combine!

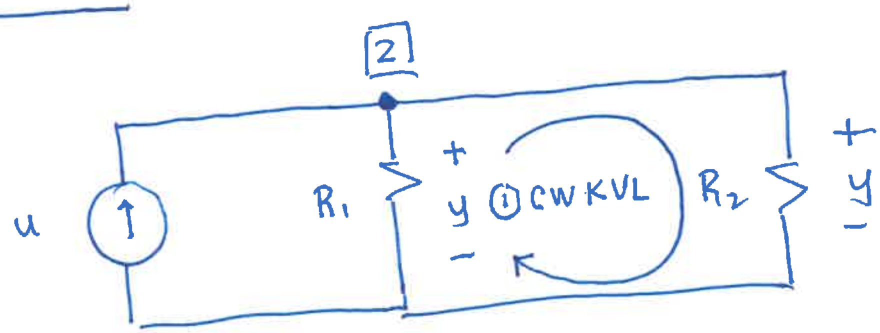
Example 5 (Current Divider)

Relate y to u

($\exists R_1, R_2$)



Solution:



2 KCL: (+ohm)

KCL, ohm, ohm

$$\boxed{u = \left(\frac{y}{R_1}\right) + \left(\frac{y}{R_2}\right)}$$

$$= y\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$\frac{1}{R_{parallel}}$

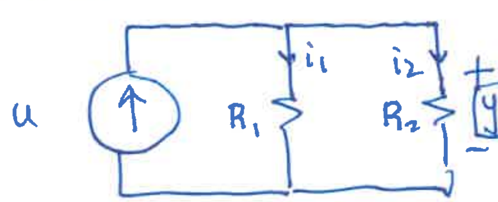
Note: R_1 & R_2 are said to be in parallel since same voltage y appears across each!

PLEASE LEARN THIS ASAP!

you do the algebra

$$\Rightarrow \boxed{R_{parallel} = R_1 \| R_2 \triangleq \frac{R_1 R_2}{R_1 + R_2}}$$

Concise Summary:



$\boxed{y = (R_1 \| R_2)\, u}$

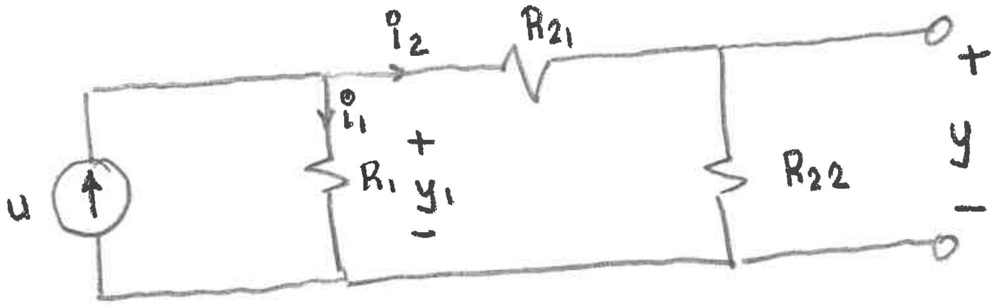
Current Divider Formulae

$$\boxed{i_1 = \frac{y}{R_1} = \left(\frac{R_2}{R_1 + R_2}\right) u}$$

$$\boxed{i_2 = \frac{y}{R_2} = \left(\frac{R_1}{R_1 + R_2}\right) u}$$

Problem 5

(Current Divider + 1 Extra Resistor)



a) Relate i_1 to u (R_1, R_{21}, R_{22}) without introducing another variable.

b) " y_1 " " " " " " " "

c) " i_2 " " " " " " " "

d) " y " " " " " " " "

The above are 4 seperate problems !!!!

Note: For each, draw a clean circuit & only use the variable specified (in addition to u, R_1, R_{21}, R_{22})

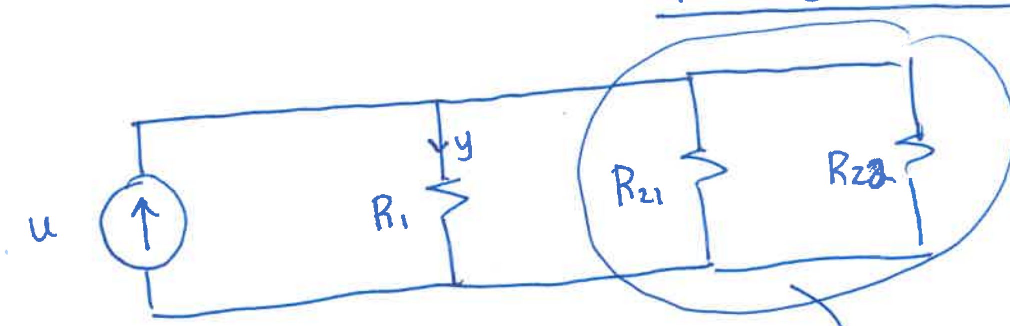
i.e use KVL, KCL, & Ohm to propagate the specified variable around the circuit so that you can get an equation that relates the variable to u & the resistors. Then solve the equation for the variable of interest.

150

Example 6

(Current Divider with 1 extra Resistor)

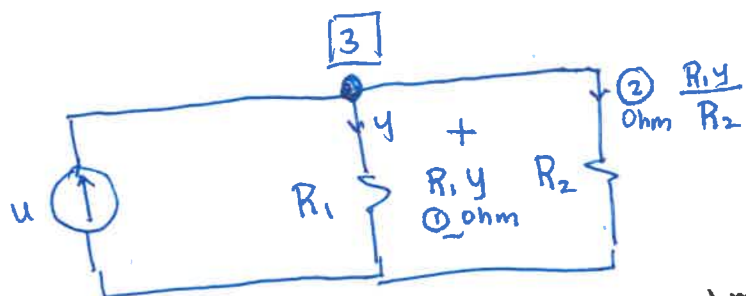
Relate y to u (& R_1, R_{21}, R_{22})



solution:

R_{21} & R_{22} are in parallel (since they have same voltage $\overset{+}{R_1 y}\,\underset{-}{}$ across them!) $\Longrightarrow$

$$R_2 \triangleq R_{21} \| R_{22} \triangleq \frac{R_{21}R_{22}}{R_{21}+R_{22}}$$



NOTE: This is circuit in Example 5!

$\boxed{3}$ KCL (+ohm): $\quad u \overset{KCL}{=} (y) + \left(\frac{R_1 y}{R_2}\right)^{\text{ohm}}$

$$= y\left[1 + \frac{R_1}{R_2}\right]$$

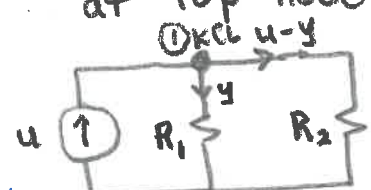
$$= y\left[\frac{R_1 + R_2}{R_2}\right]$$

$$\boxed{y = \left(\frac{R_2}{R_1 + R_2}\right)u}$$

Note: This is current i_1 in Example 5!

VERY Strong Suggestion: (MUST DO!!!)

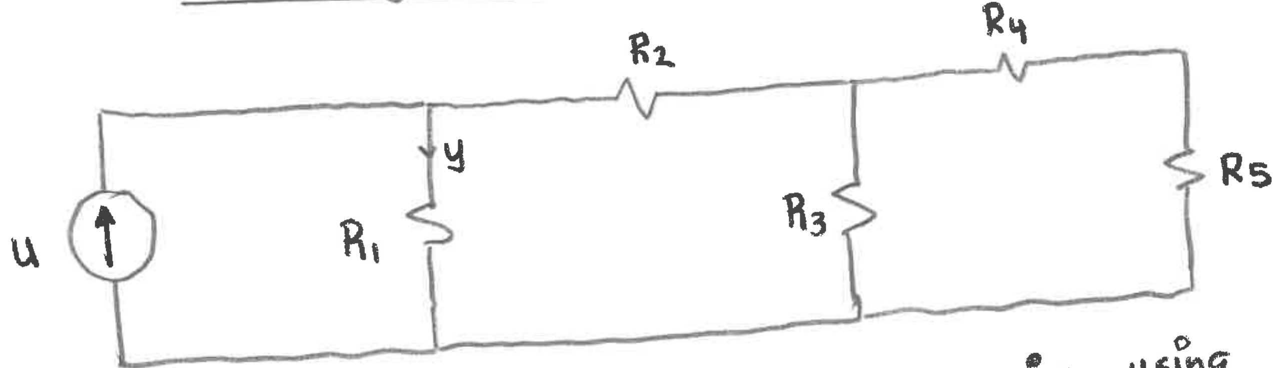
Redo this with step $\boxed{1}$ being a KCL at top node:



Note: A KVL in right loop will yield same result!

Problem 6 (Current Divider with Extra Resistors)

Relate y to u (5 resistors in circuit)



a) Do by showing that $y = \left[\dfrac{\hat{R}_2}{R_1 + \hat{R}_2} \right] u$

i.e. using series-parallel concepts to make cct look like current divider in Example 5!

where $\hat{R}_2 \triangleq R_2 + [R_3 \,||\, (R_4 + R_5)]$

b) Now relate y to u by using KCL, Ohm, KVL to propagate y rightward through the circuit

(This is VERY IMPORTANT!)

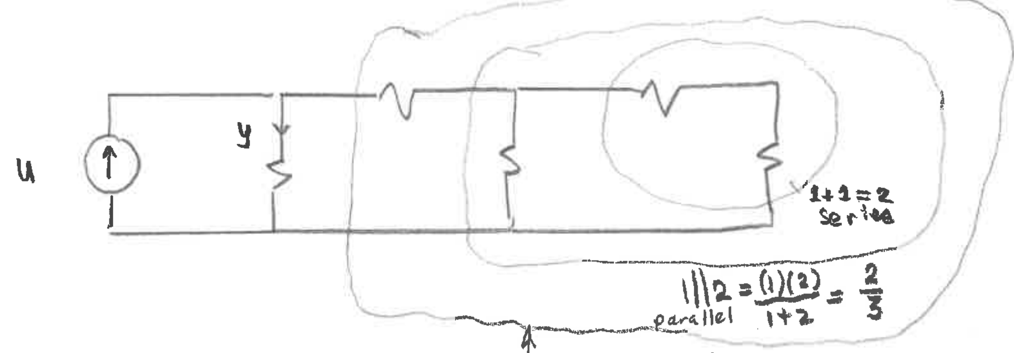
Note: (1) You MUST show that the answer from (a) & that from (b) are identical!

(2) While (a) is MUCH easier & faster, the methods of (b) are MORE FUNDAMENTAL & hence MORE IMPORTANT TO MASTER!

(3) By properly organizing your steps in (b), you can derive the nice, clean, elegant result stated in (a).

Example 6 (Problem 6 with All $R_i = 1$)

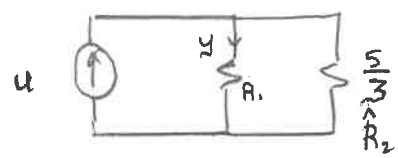
(continued)



← using series-parallel concepts

$1 \| 2 = \frac{(1)(2)}{1+2} = \frac{2}{3}$ parallel

$1 + \frac{2}{3} = \frac{5}{3}$ series



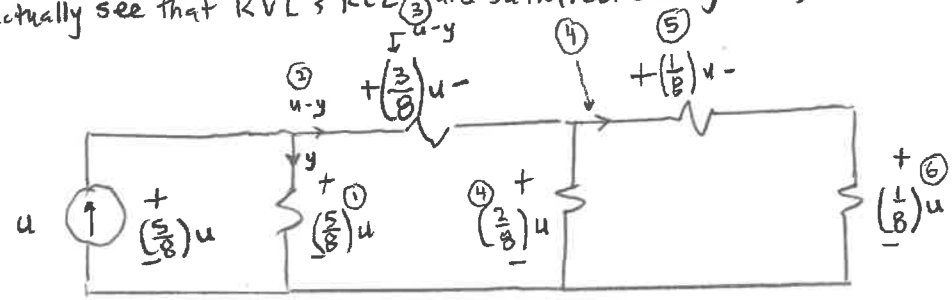
Current Division ⟹ Example 5

$$y = \left(\frac{\hat{R}_2}{R_1 + \hat{R}_2}\right) u = \left(\frac{\frac{5}{3}}{1 + \frac{5}{3}}\right) u \quad (1+\tfrac{5}{3} = 8/3)$$

$$y = \left(\frac{5}{8}\right) u$$

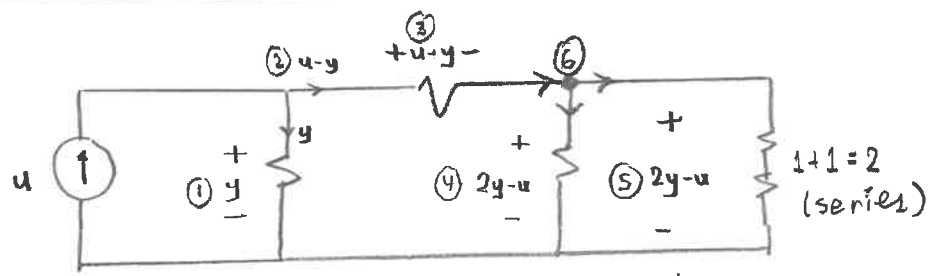
Compute all v's, i's in original circuit:

(to actually see that KVL & KCL are satisfied everywhere)



Note: KVL is satisfied within each loop & KCL is satisfied at each node!

KVL, KCL, Ohm with all $R_i = 1$



$$⑥ \text{KCL (+ohm)} = \left(\frac{u-y}{1}\right) = \left(\frac{2y-u}{1}\right) + \left(\frac{2y-u}{2}\right) \Longrightarrow 2u - 2y = 4y - 2u + 2y - u$$

$$\Longrightarrow 5u = 8y$$

$$\Longrightarrow y = \left(\frac{5}{8}\right) u \checkmark$$

Note:

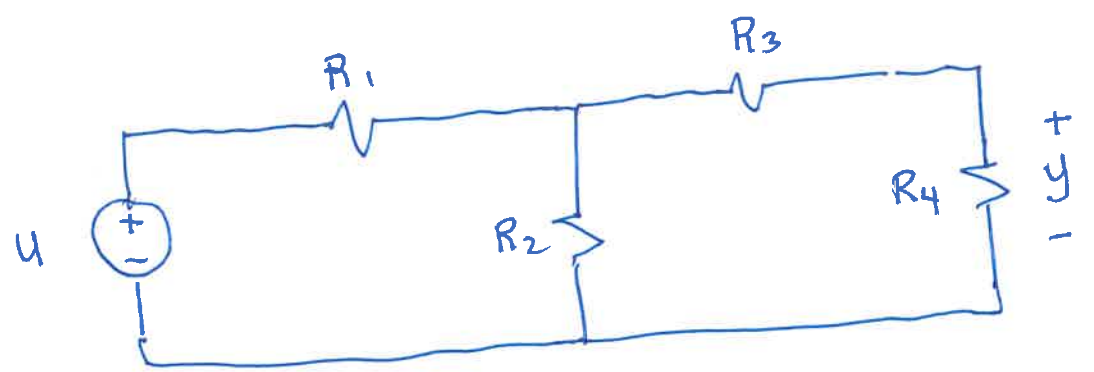
It is always useful to redo examples/problems with all $R_i = 1$ so that you can significantly elevate/strengthen your circuit strategizing!

Voltage Divider + Extra Resistors:

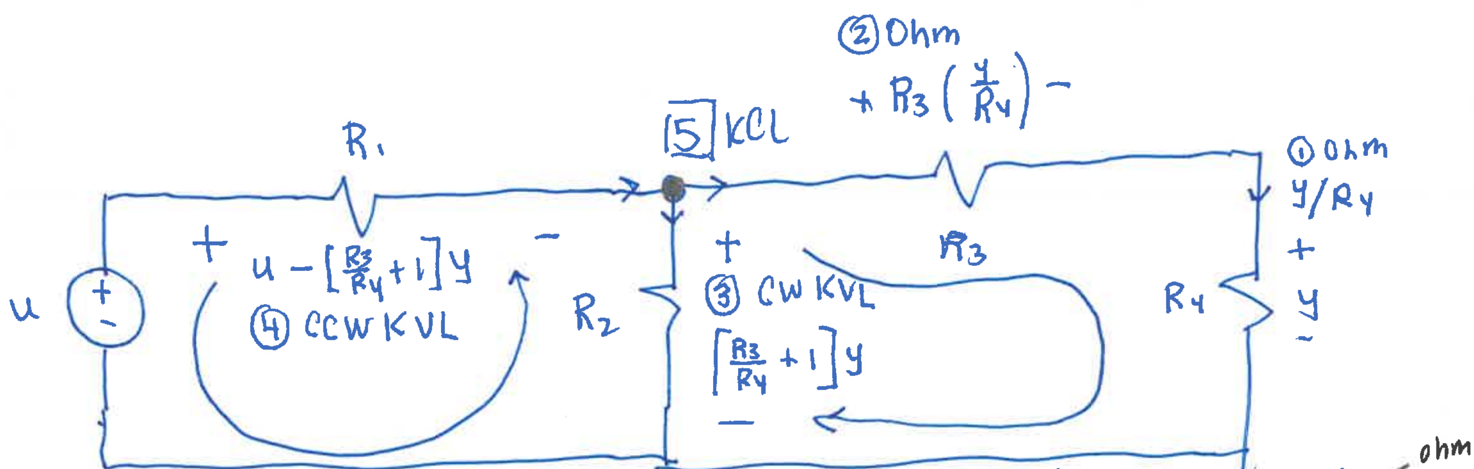
Example 7 (Propagating y leftward)

170

[9] Relate y to u



Solution: (always show all work on your circuit!)



[5] KCL = (+Ohm)

$$\boxed{\left[\frac{u-\left[\frac{R_3}{R_4}+1\right]y}{R_1}\right] \overset{\text{KCL}}{=} \left(\frac{\left[\frac{R_3}{R_4}+1\right]y}{R_2}\right) + \left(\frac{y}{R_4}\right)}$$

ohm, ohm, ohm

Always box in your eq relating y to u!!!

algebra:

$$\frac{u}{R_1} = y\left[\frac{\frac{R_3}{R_4}+1}{R_1} + \frac{\frac{R_3}{R_4}+1}{R_2} + \frac{1}{R_4}\right]$$

$$\boxed{y = \frac{1}{R_1}\left[\frac{1}{\frac{\frac{R_3}{R_4}+1}{R_1} + \frac{\frac{R_3}{R_4}+1}{R_2} + \frac{1}{R_4}}\right]u}$$

if $R_i = R$

$$= \frac{1}{R}\left[\frac{1}{\frac{2}{R}+\frac{2}{R}+\frac{1}{R}}\right]u$$

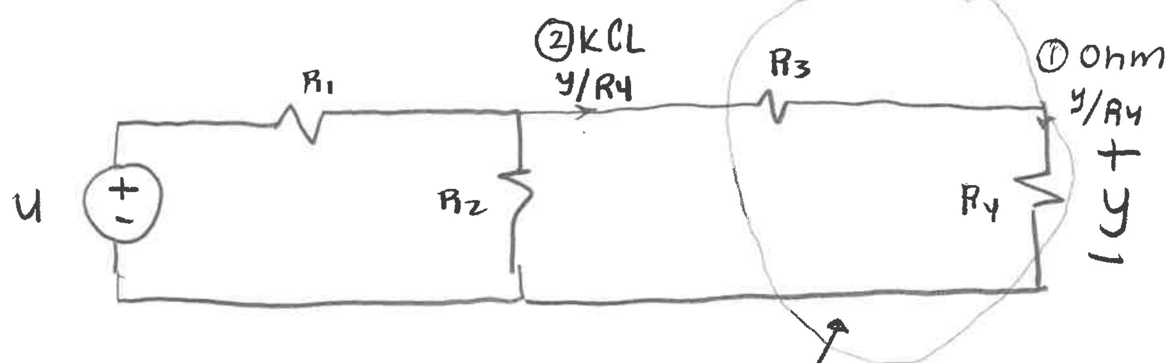
$$= \frac{1}{5}u \quad \text{for } R_i = R$$

Example 7

Use series-parallel concepts to relate y to u.

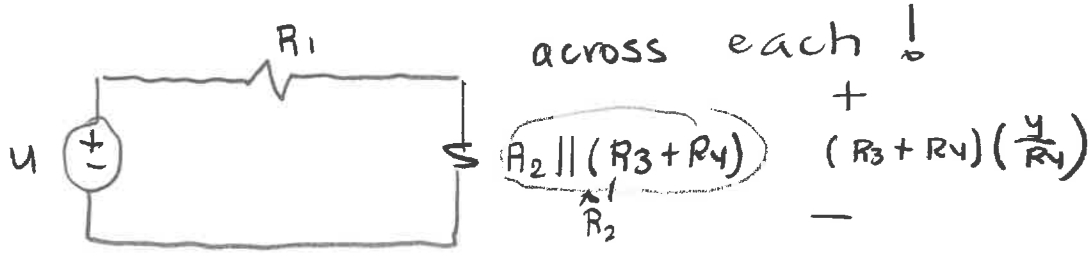
(Instead of straight KVL, KCL, Ohm as we did on pg 170)

Solution:



③ R_3 & R_4 are in series with current y/R_4 passing through each!

⑥ R_2 & (R_3+R_4) are in parallel with voltage $(R_3+R_4)\left(\frac{y}{R_4}\right)$ across each!



By voltage division (& KVL), it follows that voltage across $\hat{R}_2$ is:

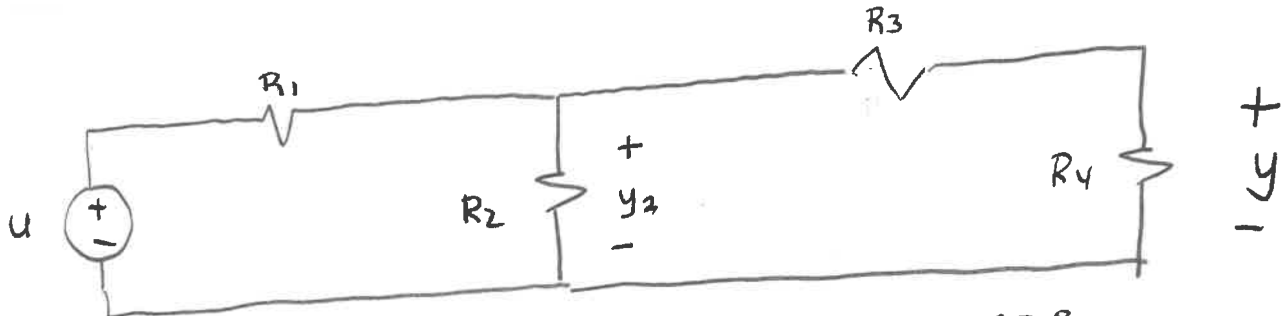
$$\boxed{(R_3+R_4)\left(\frac{y}{R_4}\right) = \left(\frac{\hat{R}_2}{R_1+\hat{R}_2}\right)u = \left[\frac{R_2 \| (R_3+R_4)}{R_1+\left[R_2 \| (R_3+R_4)\right]}\right]u}$$

Example 7

Algebra: From previous expression, we get the MUCH cleaner (simpler) looking result:

$$\boxed{y = \left[\frac{R_4}{R_3 + R_4}\right] \underbrace{\left[\frac{R_2 \,\|\, (R_3 + R_4)}{R_1 + [R_2 \,\|\, (R_3 + R_4)]}\right]}_{\text{call this } y_2} u}$$

Lets visualize this on our circuit:



Note: By voltage division

$$y = \left(\frac{R_4}{R_3 + R_4}\right) y_2$$

where y_2 is voltage across parallel combination

$$\hat{R}_2 = R_2 \,\|\, (R_3 + R_4)$$

i.e.

$$y_2 = \left[\frac{\hat{R}_2}{R_1 + \hat{R}_2}\right] u$$

$$= \left[\frac{R_2 \,\|\, (R_3 + R_4)}{R_1 + (R_2 \,\|\, (R_3 + R_4))}\right] u$$

as on pg 170

Note: This shows that the above result is readily (quickly) obtained by inspection using series-parallel & voltage division concepts!

$\boxed{\text{CHECK}}$: letting $R_i = 1$ yields

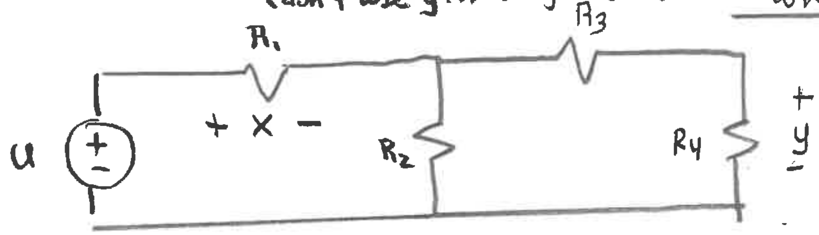
Always good to do a CHECK!

$$y = \left[\frac{1}{1+1}\right] \left[\frac{1\|2}{1 + 1\|2}\right] u = \left[\frac{1}{2}\right] \left[\frac{2/3}{1 + 2/3}\right] u = \left(\frac{1}{2}\right)\left(\frac{2}{5}\right) u = \left(\frac{1}{5}\right) u \checkmark$$

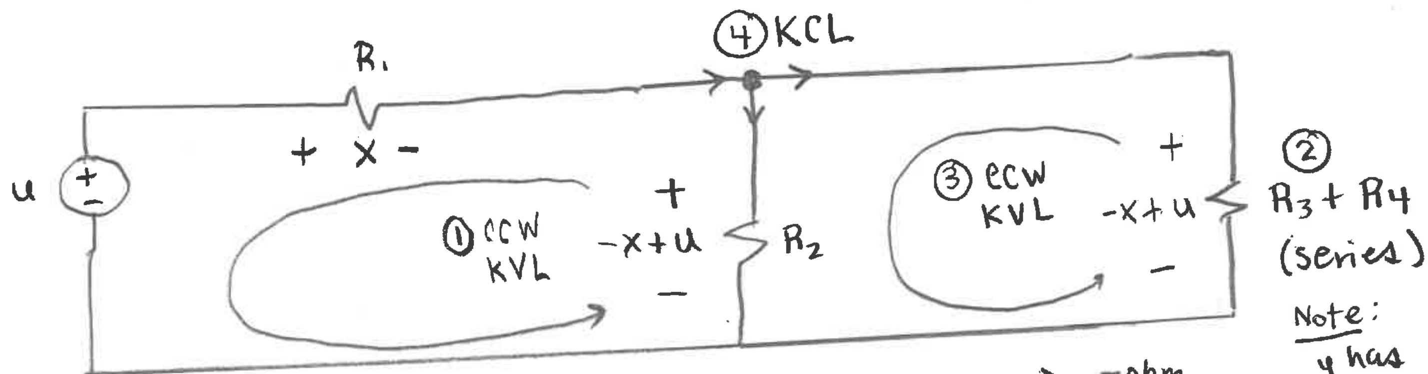
$1 + 2/3 = 5/3$

Example 7 (Propagating x rightward)

b Relate x to u (don't use y... only x !!!) then solve for x & y when $R_i = R$



Solution: (always show all work on your circuit!)



Note: y has been lost here!!!

$$④\ \text{KCL} = \boxed{\left(\frac{x}{R_1}\right) = \left(\frac{-x+u}{R_2}\right) + \left(\frac{-x+u}{R_3+R_4}\right)}$$

(+ohm)

ohm, KCL, ohm, ohm

↑ Always box in your eq relating x to u!

algebra:
$$x\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3+R_4}\right] = u\left[\frac{1}{R_2}+\frac{1}{R_3+R_4}\right]$$

$$\Rightarrow \boxed{x = \frac{\left[\frac{1}{R_2}+\frac{1}{R_3+R_4}\right]}{\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3+R_4}\right]}\,u} \underset{R_i=R}{=} \frac{\left[\frac{1}{R}+\frac{1}{2R}\right]}{\left[\frac{1}{R}+\frac{1}{R}+\frac{1}{2R}\right]}u$$

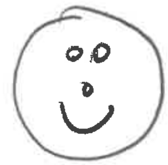
$$= \left(\frac{\frac{2+1}{2R}}{\frac{2+2+1}{2R}}\right)u$$

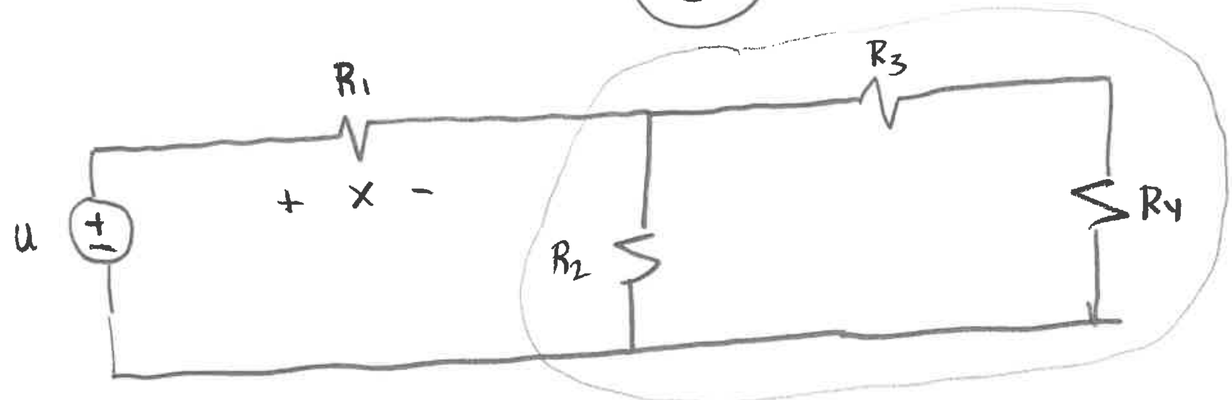
$$\boxed{x = \left(\frac{3}{5}\right)u}$$

↑ $R_i = R$

Example 7

Here is a MUCH EASIER way to solve for x ---- using series-parallel voltage-divider concepts!



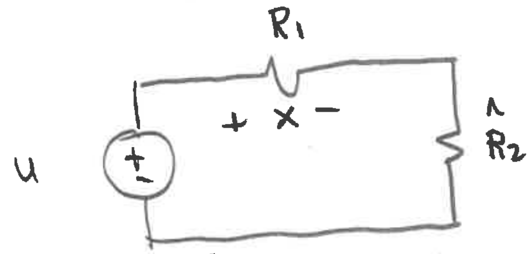


We note that all this can be replaced by:

$$\hat{R}_2 = R_2 \parallel (R_3 + R_4)$$

because R_3 & R_4 are in series & R_2 is in parallel with (R_3+R_4).

This then yields the voltage divider circuit:



$\Longrightarrow$ Now (from Example 2), we get

$$x = \left[\frac{R_1}{R_1 + \hat{R}_2}\right] u$$

or

This is BEST way to write x!

$$\boxed{x = \left[\frac{R_1}{R_1 + R_2 \parallel (R_3+R_4)}\right] u}$$

For $R_i = R$, $x = \left[\frac{R}{R + R\|2R}\right] u$

$= \left[\frac{R}{R + \frac{2}{3}R}\right] u = \left(\frac{3}{5}\right) u$ ✓✓✓

Example 7

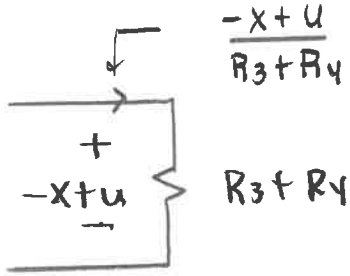
Check on x:

x must be

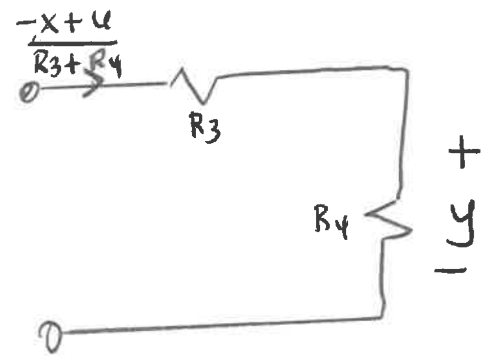
$$x = u - \left[\frac{R_3}{R_4}+1\right]y$$

Now that we have $x \overset{R_i = R}{=} \left(\frac{3}{5}\right)u$

How do we get y?



← this current passes through R_3 & R_4:



From this, it follows that

$$y \overset{\text{Ohm}}{=} R_4\left[\frac{-x+u}{R_3+R_4}\right]$$

Letting $R_i = R$ then yields:

$$y = R\left[\frac{-x+u}{2R}\right]$$

$$= \frac{1}{2}\left[u - x\right]$$

$$= \frac{1}{2}\left[u - \left(\frac{3}{5}\right)u\right]$$

$$= \frac{1}{2}\left[\frac{2}{5}u\right]$$

$$\boxed{y = \left(\frac{1}{5}\right)u}$$

Check on x:

x must be

$$x = u - \left[\frac{R_3}{R_4}+1\right]y$$

$$\overset{R_i=R}{=} u - [2y]$$

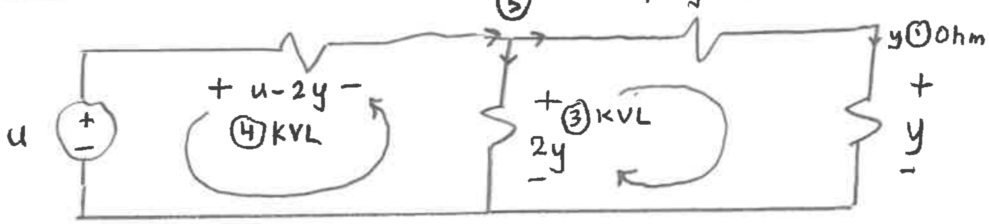
$$= u - 2\left(\frac{1}{5}\right)u$$

$$\boxed{x = \frac{3}{5}u}$$

which agrees with what we got above!

Example 7 (Summary for all $R_i = 1$)

KVL, KCL, Ohm:

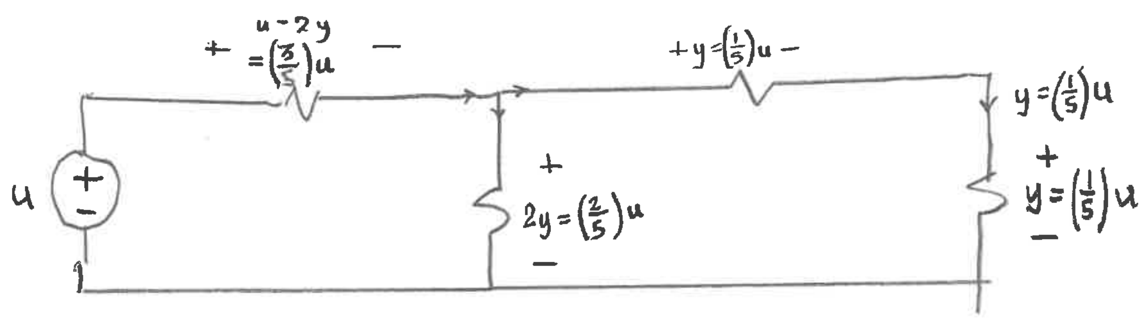


5 KCL (+ Ohm) = $\left(\frac{u-2y}{1}\right) = \left(\frac{2y}{1}\right) + y \xRightarrow{\text{algebra}} u = 5y$

$\boxed{y = \left(\tfrac{1}{5}\right)u}$

Compute all v's & i's from above analysis:

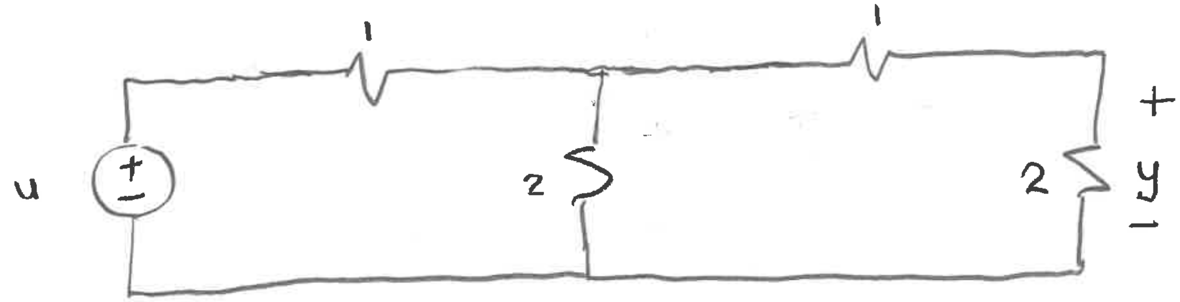
(so that you can see that KVL & KCL are satisfied everywhere)



Note: KVL is infact satisfied around each loop!

KCL " " " " at each node!!

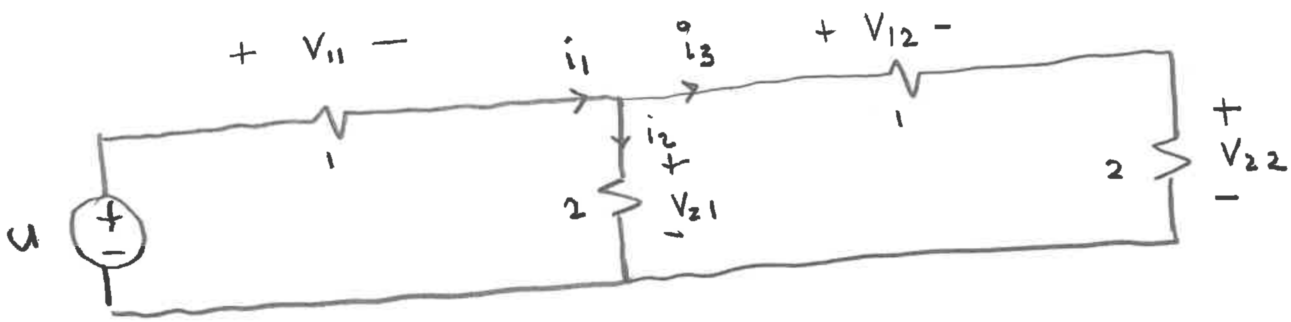
Problem 7 (Voltage Divider with Extra Resistors)



a) Relate y to u by using Ohm, KCL, KVL to propagate y leftward to get an equation relating y to u & the resistors in the circuit.

<u>Hint:</u> Duplicate steps delineated in Example 7!

b) Given the above, compute each of the voltages & currents in the circuit (in terms of u!!!):



c) Show that

$u = V_{11} + V_{21}$ (i.e. KVL satisfied in left loop)

$V_{21} = V_{12} + V_{22}$ (i.e. KVL satisfied in right loop)

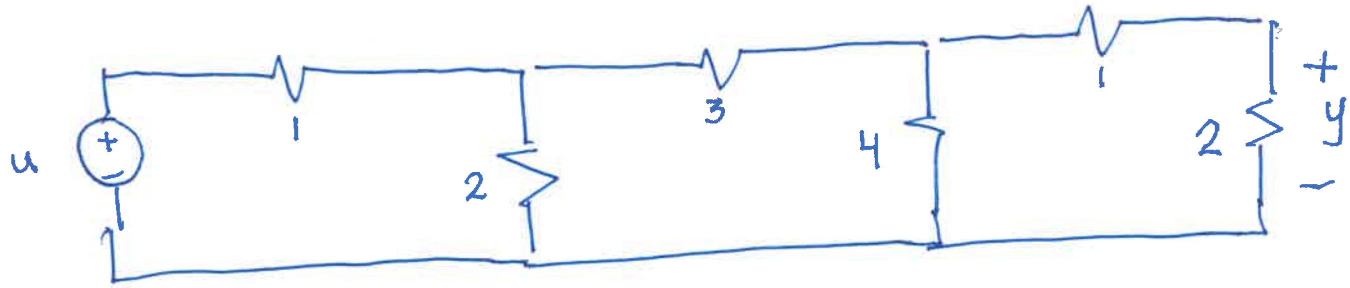
$i_1 = i_2 + i_3$ (KCL satisfied at top node)

[3

Example 8

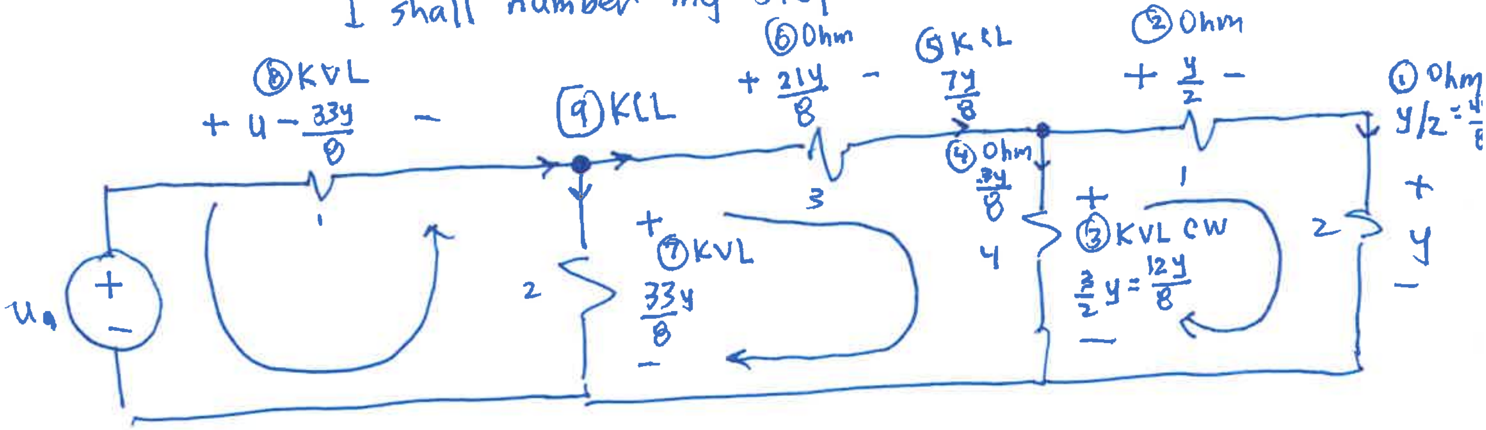
(Propagating y leftward)

Relate y to u



Solution: (You must always show all work on your circuit!)

I shall number my steps:



$\boxed{9}$ KCL: (+ Ohm)

$$\boxed{\frac{\left(u-\frac{33y}{8}\right)}{1} = \left(\frac{\frac{33y}{8}}{2}\right) + \left(\frac{7y}{8}\right)}$$

ohm — KCL — ohm

algebra

$$u = y\left[\underbrace{\frac{33}{8}}_{\frac{66}{16}} + \frac{33}{16} + \underbrace{\frac{7}{8}}_{\frac{14}{16}}\right]$$

$$= y\left[\frac{113}{16}\right]$$

$$\boxed{y = \left(\frac{16}{113}\right)u}$$

Note: Voltage x across top left 1Ω resistor is

$$x = u - \frac{33y}{8} = u - \left(\frac{33}{8}\right)\left(\frac{16}{113}\right)u$$

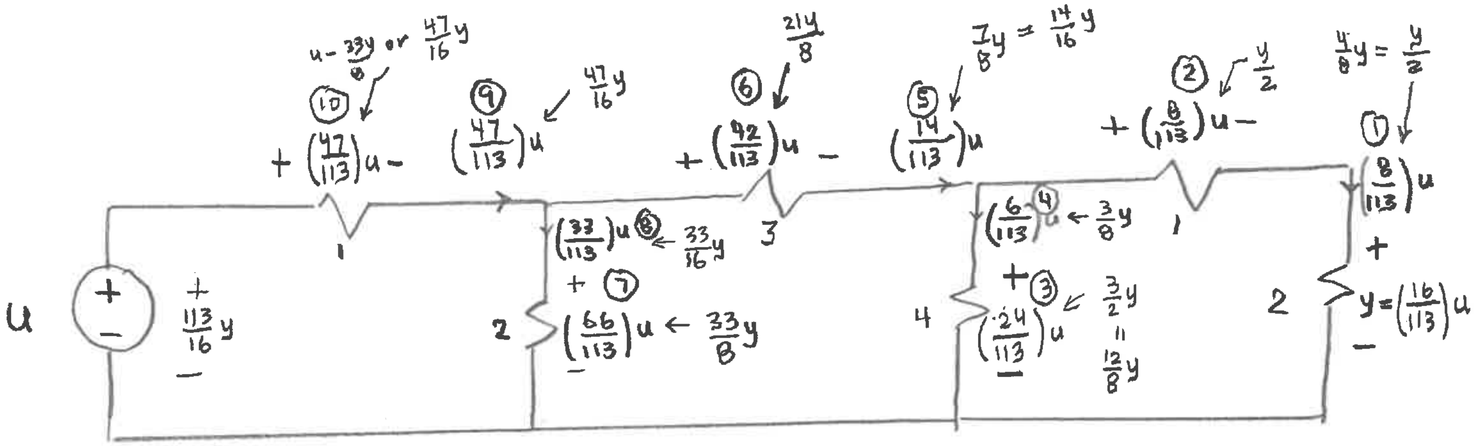
$$= u - \frac{66}{113}u$$

$$\boxed{x = \left(\frac{47}{113}\right)u}$$

Example 8

(Summary for all $R_i = 1$)

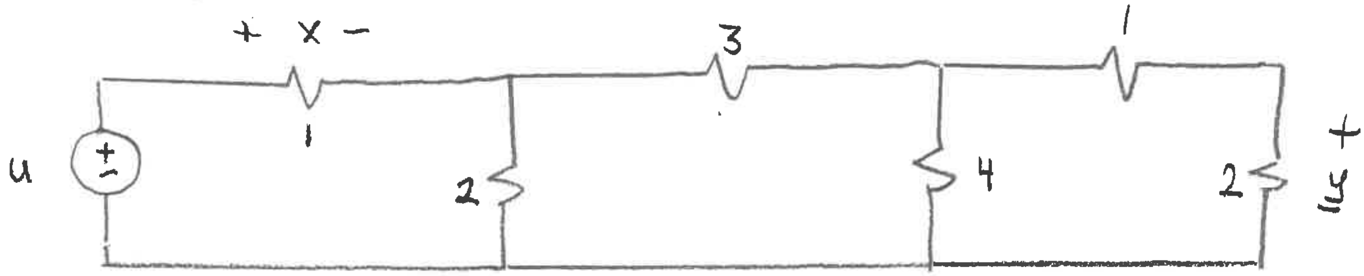
Compute all v's & i's to actually see that KVL is satisfied within each loop & KCL is satisfied at each node:



Note: KVL is in fact satisfied within each loop!
KCL " " " " at each node!

Problem 8

a) Relate x to u by propagating x rightward through the circuit. (Only use variable x --- NOT y!)



Note: The x you get must agree with that found in Example 8 !!!

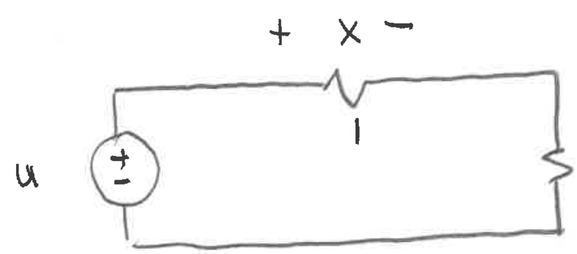
b) Now determine y

Note: The y you get must agree with that found in Example 8 !!!

c) Now find x by using series-parallel concepts to note that the above circuit may be written as

PLEASE LEARN!

(master series-parallel concepts !!!)



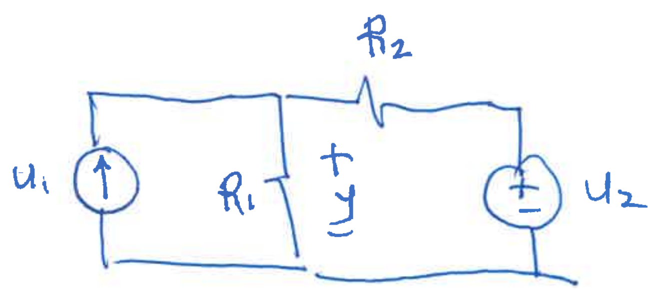
$$\hat{R}_2 = 2 \,\|\, \left\{ 3 + \left[4 \,\|\, (1+2) \right] \right\}$$

Note: Result you get for x here must agree with that found in (a)!

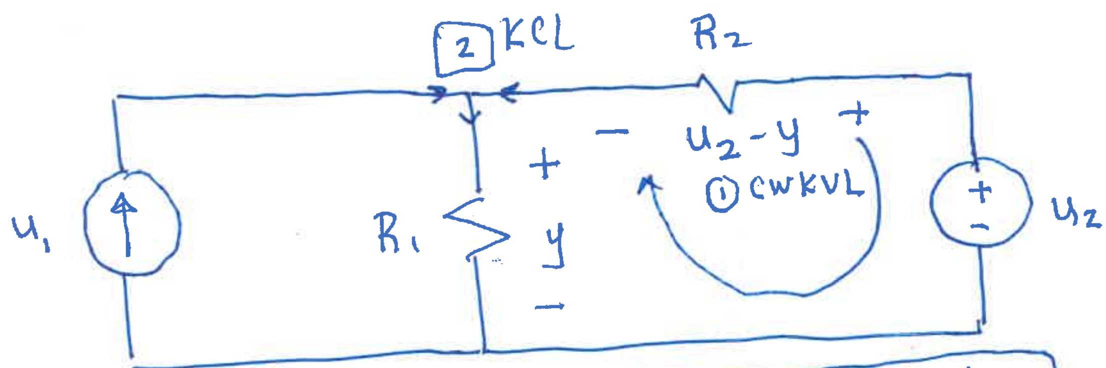
Example 9 (2 Sources & Superposition) 240

Relate y to u_1 & u_2

(& R_1, R_2)



solution: (You MUST always show all work on your circuit!)



2 KCL = (+ohm)

$$\boxed{u_1 \overset{\text{KCL}}{=} \underbrace{\left(\frac{y}{R_1}\right)}_{\text{ohm}} + \underbrace{\left(\frac{u_2 - y}{R_2}\right)}_{\text{ohm}}}$$

algebra: $u_1 + \dfrac{u_2}{R_2} = y\left[\dfrac{1}{R_2} + \dfrac{1}{R_1}\right]$

$$R_2 u_1 + u_2 = y\left[1 + \frac{R_2}{R_1}\right]$$

$$= y\left[\frac{R_1 + R_2}{R_1}\right]$$

$$\boxed{y = \left[\frac{R_1 R_2}{R_1 + R_2}\right] u_1 + \left[\frac{R_1}{R_1 + R_2}\right] u_2}$$

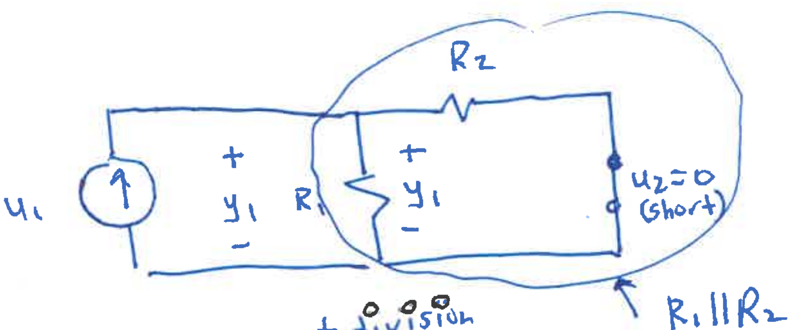
$$\overset{R_i = 1}{=} \left[\tfrac{1}{2}\right](u_1 + u_2)$$

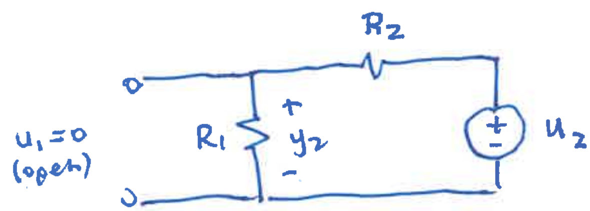
$$= \tfrac{1}{2} u_1 + \tfrac{1}{2} u_2$$

Example 9 – (Superposition: A Consequence of Linearity)

Lets show that $\quad y = y_1 + y_2$

where $\quad y_1 = y\big|_{u_2=0} \qquad y_2 = y\big|_{u_1=0}$





current division

$$\frac{y_1}{R_1} = \left(\frac{R_2}{R_1+R_2}\right)u_1$$

or

$$\boxed{y_1 = \left(\frac{R_1R_2}{R_1+R_2}\right)u_1}$$

$R_1 \| R_2$

Ohm's Law

$$\boxed{y_2 = \left(\frac{R_1}{R_1+R_2}\right)u_2}$$

Voltage Division

$\Rightarrow$ We've shown that

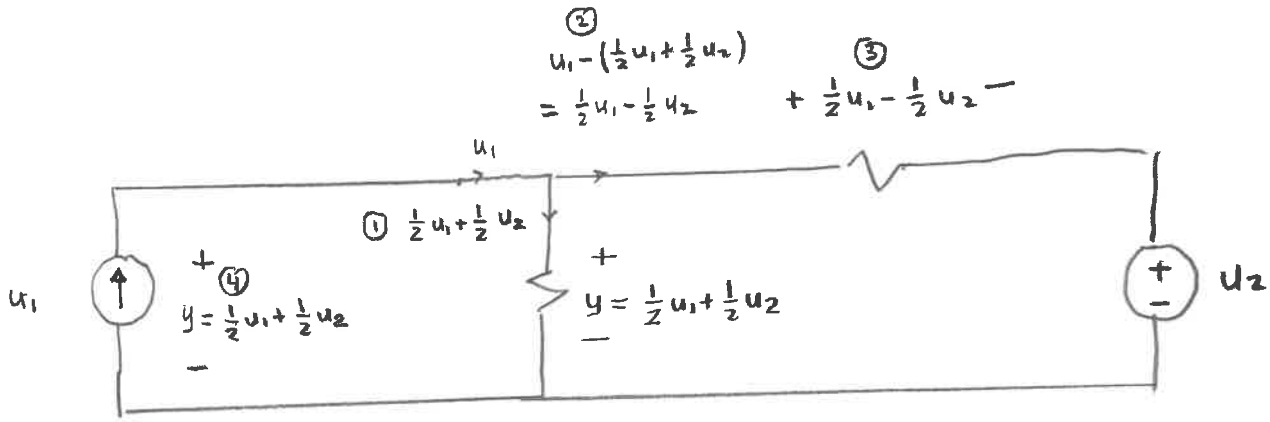
$$y = y_1 + y_2$$

$$= (R_1 \| R_2)u_1 + \left(\frac{R_1}{R_1+R_2}\right)u_2$$

Hence we have the principle of Superposition!

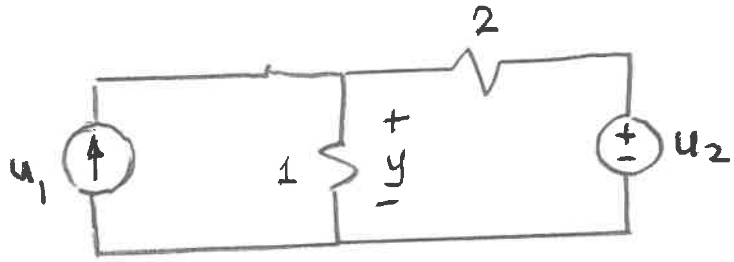
Example 9 (Summary for all $R_i = 1$)

Compute all v's & i's for $R_i = 1$:



Note: KVL is in fact satisfied within each loop!
KCL " " " " at each node!!

$\boxed{\text{Problem 9}}$ (2 sources & design)



$\boxed{a}$ Relate y to u_1 & u_2 (do NOT introduce another variable!!! use y, u_1, u_2)

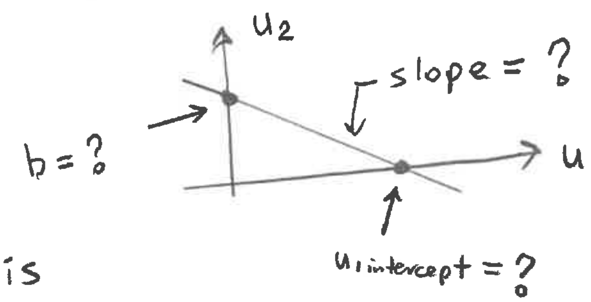
$\boxed{b}$ Suppose we desire $y = \frac{1}{3}$.

What equation must u_1 & u_2 satisfy to yield $y = \frac{1}{3}$?

---- Write the equation as

$$u_2 = \boxed{?} + \boxed{?}\, u_1$$

Now graph u_2 vs u_1



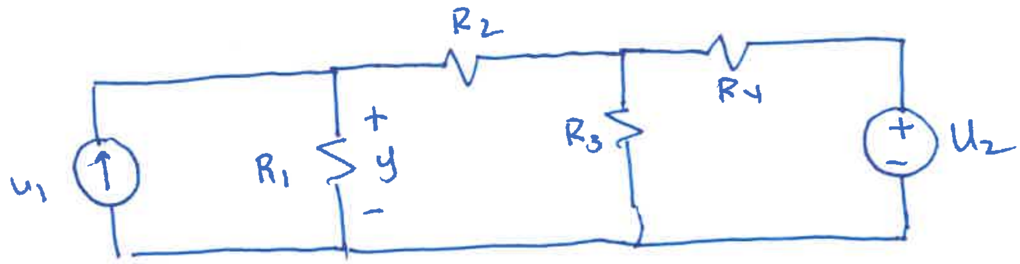
Note:

Here, we have used our analysis in order to figure out the set of all u_1-u_2 combinations that result in $y = \frac{1}{3}$.

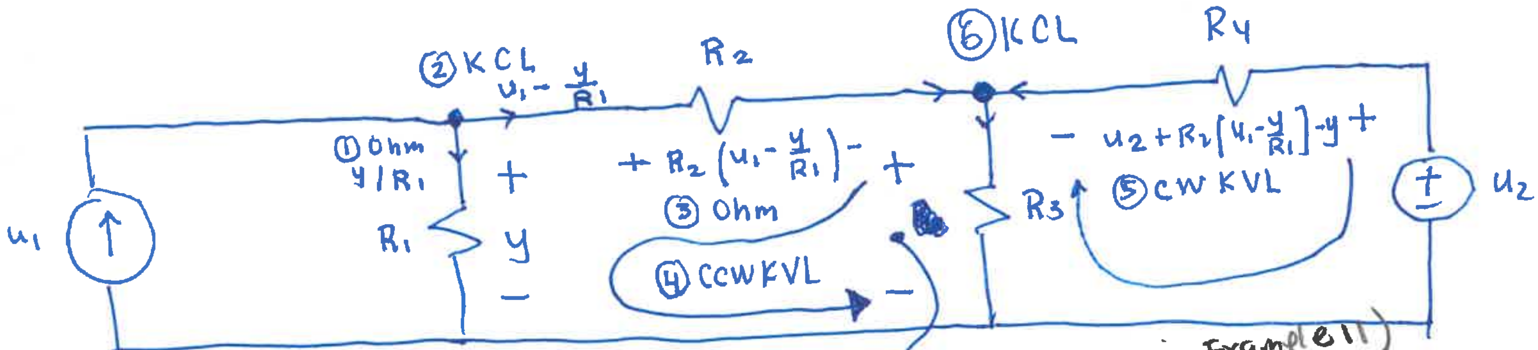
Given this, this is a design problem!

Example 10 (2 Sources & Superposition Again)

a Relate y to u (& $R_{1,2,3,4}$) (do NOT introduce another variable... use y!!!)



Solution: (You must always show all work on your circuit!)



④ $-R_2\left[u_1 - \frac{y}{R_1}\right] + y$ (= w_2 in Example 11)

⑥ KCL = (+ ohm)

$$\boxed{\left(u_1 - \frac{y}{R_1}\right) + \left(\frac{u_2 + R_2\left[u_1 - \frac{y}{R_1}\right] - y}{R_4}\right) = \frac{-R_2\left[u_1 - \frac{y}{R_1}\right] + y}{R_3}}$$

KCL

Ohm Ohm

algebra:
$$u_1\left[1 + \frac{R_2}{R_4} + \frac{R_2}{R_3}\right] + \frac{u_2}{R_4} = y\left[\frac{1}{R_1} + \frac{R_2/R_1}{R_4} + \frac{1}{R_4} + \frac{R_2/R_1}{R_3} + \frac{1}{R_3}\right]$$

$R_i = 1$:
$$y = \left[\frac{1+2}{1+1(2)+1(2)}\right]u_1 + \left[\frac{1}{1+1(2)+1(2)}\right]u_2 = \left(\frac{3}{5}\right)u_1 + \left(\frac{1}{5}\right)u_2$$

$$\boxed{y = \left[\frac{1 + R_2\left(\frac{1}{R_3} + \frac{1}{R_4}\right)}{\frac{1}{R_1} + \frac{1}{R_3}\left(\frac{R_2}{R_1} + 1\right) + \frac{1}{R_4}\left(1 + \frac{R_2}{R_1}\right)}\right]u_1 + \frac{1}{R_4}\left[\frac{1}{\frac{1}{R_1} + \frac{1}{R_3}\left(\frac{R_2}{R_1} + 1\right) + \frac{1}{R_4}\left(1 + \frac{R_2}{R_1}\right)}\right]u_2}$$

$= y_1 + y_2$

call this y_1 call this y_2

Example 10

algebra on y_1:

$$\frac{y_1}{u_1} = \frac{1 + \dfrac{R_2}{R_3 \| R_4}}{\dfrac{1}{R_1} + \dfrac{1}{R_3 \| R_4}\left(\dfrac{R_2}{R_1} + 1\right)}$$

$$= \frac{R_2 + R_3 \| R_4}{\dfrac{R_3 \| R_4}{R_1} + \dfrac{R_2}{R_1} + 1}$$

$$= \frac{R_1 \left[R_2 + R_3 \| R_4 \right]}{R_1 + \left[R_2 + R_3 \| R_4 \right]}$$

$$= R_1 \, \| \, (R_2 + R_3 \| R_4)$$

↑

We shall get this on next page by using superposition to find y

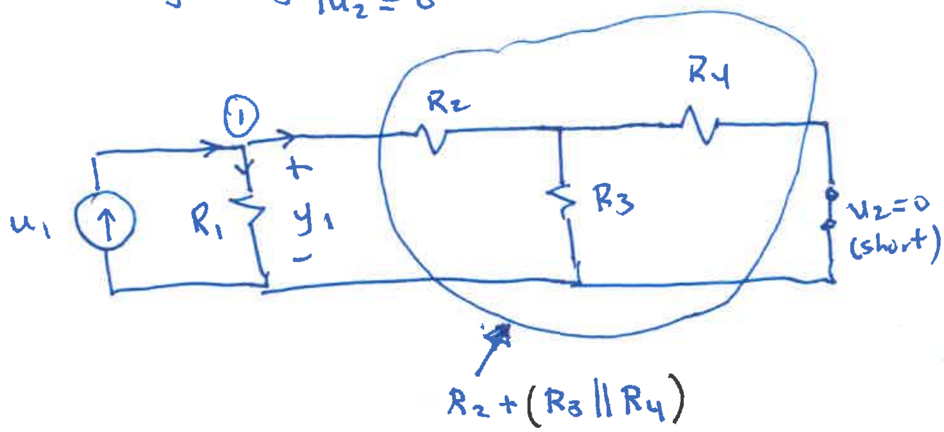
i.e. find $y_1 = y \big|_{u_2 = 0}$ & $y_2 = y \big|_{u_1 = 0}$

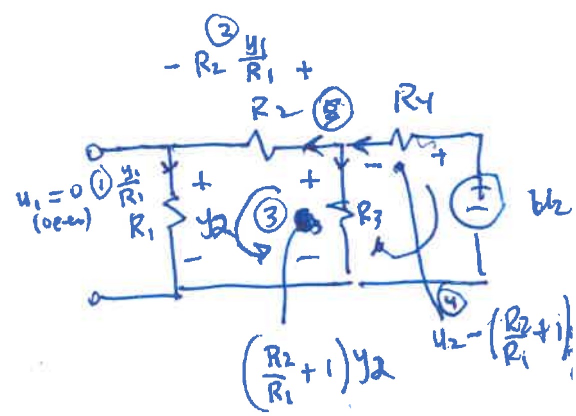
Example 10

b Now lets compute y_1 & y_2 separately showing that $y = y_1 + y_2$ by the principle of superposition!

$y_2 = y \big|_{u_1 = 0}$

$y_1 = y \big|_{u_2 = 0}$





① KCL: $u_1 = \frac{y_1}{R_1} + \frac{y_1}{R_2 + (R_3 \| R_4)}$

$$\boxed{y_1 = \left[\frac{1}{\frac{1}{R_1} + \frac{1}{R_2 + (R_3 \| R_4)}}\right] u_1}$$

Note: For $R_i = 1$ $\quad y_1 = [1 \| (1 + \tfrac{1}{2})] u_1 = [1 \| \tfrac{3}{2}] u_1 = \tfrac{3}{5} u_1$

$$\boxed{y_1 = [R_1 \| [R_2 + (R_3 \| R_4)]] u_1}$$

This is the BEST way to write y_1!

(much simpler than what we got on pg 270!)

⑤ KCL:

$$\boxed{\frac{u_2 - \left(\frac{R_2}{R_1}+1\right)y_2}{R_4} = \frac{\left(\frac{R_2}{R_1}+1\right)y_2}{R_3} + \frac{y_1}{R_1}}$$

$$\frac{u_2}{R_4} = y_2 \left[\frac{\frac{R_2}{R_1}+1}{R_4} + \frac{\frac{R_2}{R_1}+1}{R_3} + \frac{1}{R_1}\right]$$

$$\boxed{y_2 = \frac{1}{R_4}\left[\frac{1}{\frac{1}{R_1} + \left(\frac{1}{R_3} + \frac{1}{R_4}\right)\left(\frac{R_2}{R_1}+1\right)}\right] u_2}$$

$\underset{R_i = 1}{=} 1\left[\frac{1}{1 + 2(1+1)}\right] u_2 = \frac{1}{5} u_2$ agrees with what we got earlier! (on page 270)

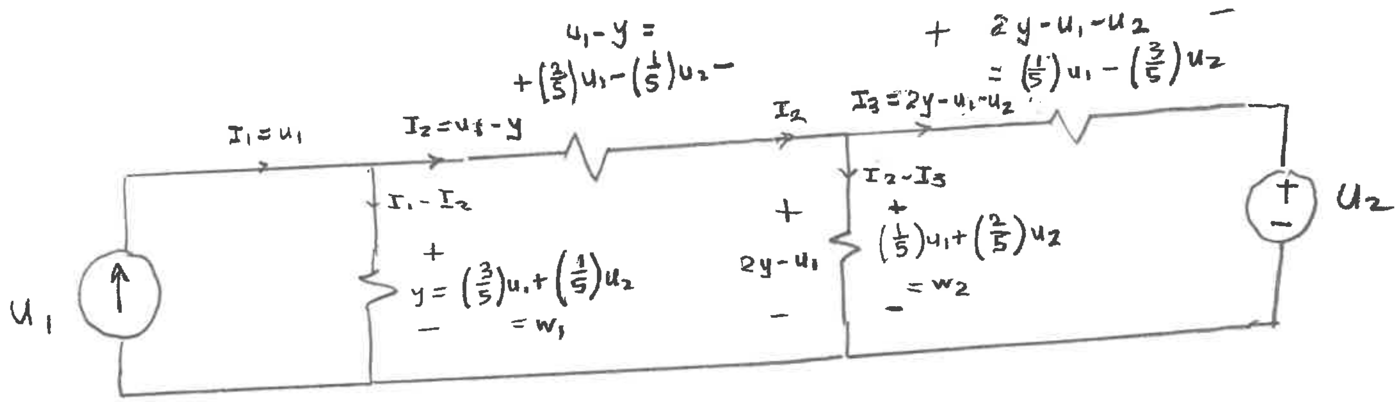
Note: The KVL, KCL, & Ohm we did on pg 270 is still VERY VERY IMPORTANT TO LEARN!

Note: This is $\frac{1}{R_3 \| R_4} = \frac{R_3 + R_4}{R_3 R_4}$

Example 10 (Summary for all $R_i = 1$)

Compute all v's & i's when all $R_i = 1$

(so that we can actually see that KVL & KCL are satisfied everywhere)



Note On Symbols:

y - used in Example 10 (page 270)

w_1, w_2 - used in Example 11 (page 310)

I_1, I_2, I_3 - used in Example 12 (page 340)

$2y - u_1 - u_2$

$y = w_1 = I_1 - I_2$
$= \left(\frac{3}{5}\right)u_1 + \left(\frac{1}{5}\right)u_2$

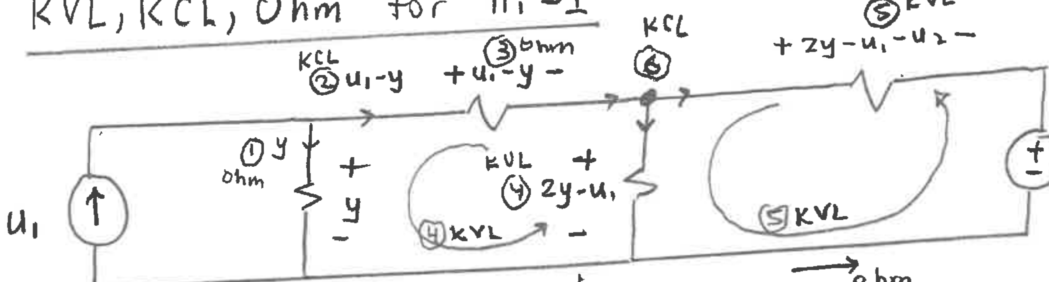
$2y - u_1 = w_2 = I_2 - I_3$

$\boxed{I_1 = u_1}$

$I_2 = w_1 - w_2$
$= u_1 - y$
$\boxed{I_2 = \left(\frac{2}{5}\right)u_1 - \left(\frac{1}{5}\right)u_2}$

$I_3 = w_2 - u_2$
$= 2y - u_1 - u_2$
$\boxed{I_3 = \left(\frac{1}{5}\right)u_1 - \left(\frac{3}{5}\right)u_2}$

KVL, KCL, Ohm for $R_i = 1$



Note:
It's always useful to solve examples/problems with all $R_i = 1$ so that you can strengthen your circuit strategizing!!!

⑥ KCL (+Ohm) $= \left(\frac{u_1 - y}{1}\right) = \left(\frac{2y - u_1}{1}\right) + \left(\frac{2y - u_1 - u_2}{1}\right) \Rightarrow 3u_1 + u_2 = 5y$

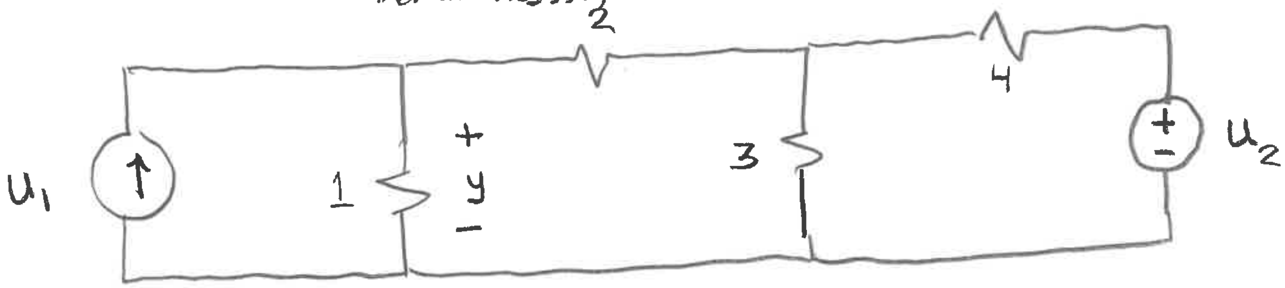
$\Rightarrow \boxed{y = \left(\frac{3}{5}\right)u_1 + \left(\frac{1}{5}\right)u_2 = w_1}$ ✓

$\boxed{w_2 = 2y - u_1 = \left(\frac{1}{5}\right)u_1 + \left(\frac{2}{5}\right)u_2}$

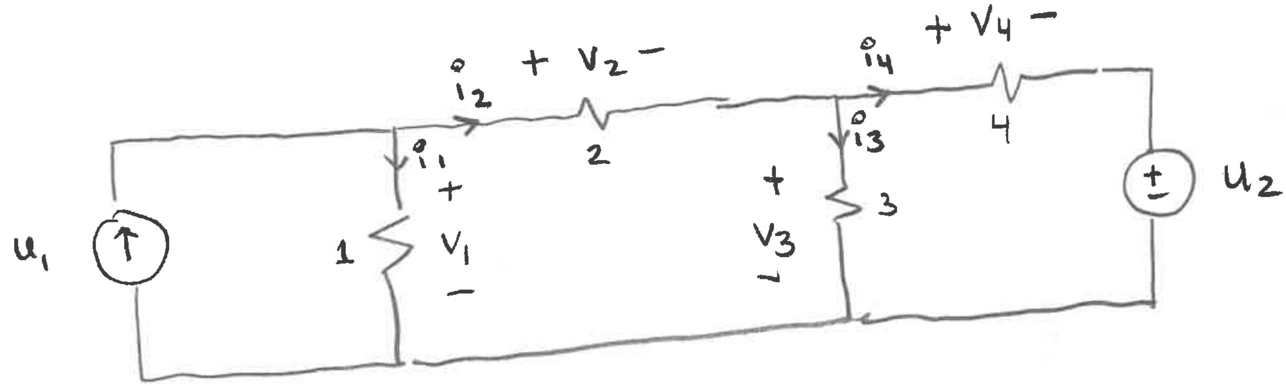
Problem 10

[a] Relate y to u_1 & u_2 (by propagating y rightward as was done in Example 10 [a])

(use only y... don't introduce any other variables!!!)



[b] Use [a] to compute each voltage & current in the circuit:



[c] Show that

$u_1 = i_1 + i_2$ (KCL satisfied at top left node)

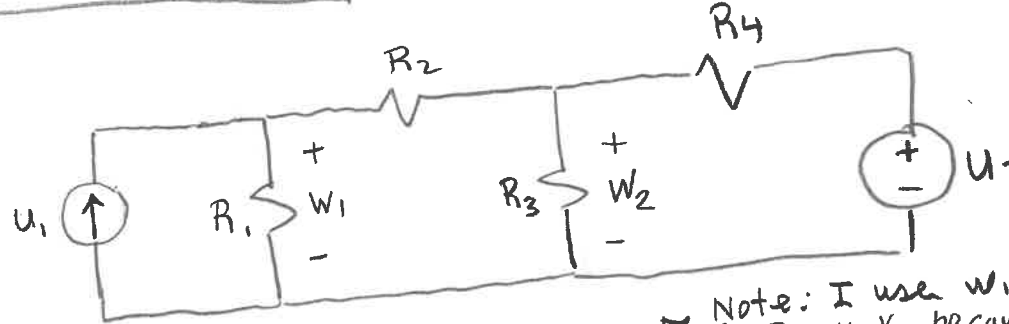
$v_1 = v_2 + v_3$ (KVL satisfied in middle loop)

$v_3 = v_4 + u_2$

Note: Try not to confuse the v_i & the u_i!

Example 11

(Example 10 via Nodal Analysis)

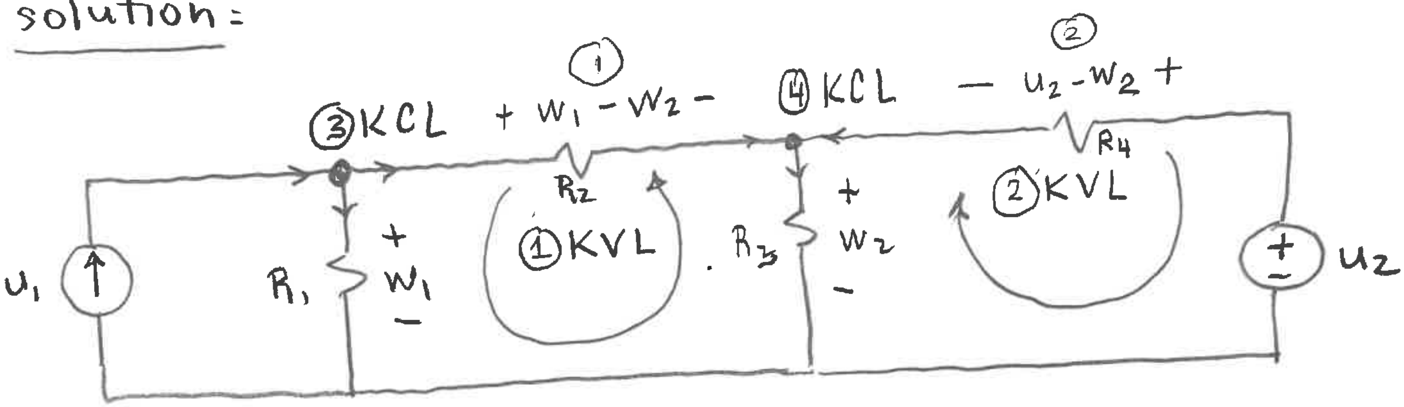


Note: I use w_1, w_2 instead of v_1, v_2 because v_1, v_2 can be confused with u_1, u_2.

Relate the nodal voltages w_1, w_2 to u_1, u_2 (& $R_{1,2,3,4}$)

(Write in "matrix - vector" form)

solution:



③ KCL (+ohm):
$$\boxed{u_1 \overset{KCL}{=} \overset{ohm}{\left(\frac{w_1}{R_1}\right)} + \overset{ohm}{\left(\frac{w_1 - w_2}{R_2}\right)}} \qquad \boxed{A}$$

④ KCL (+ohm):
$$\boxed{\overset{ohm}{\left(\frac{w_1 - w_2}{R_2}\right)} + \overset{ohm}{\left(\frac{u_2 - w_2}{R_4}\right)} \overset{KCL}{=} \overset{ohm}{\left(\frac{w_2}{R_3}\right)}} \qquad \boxed{B}$$

This gives us 2 eqs in the 2 unknowns w_1, w_2 !!! ☺

Note: From Example 10 (pg 270), it follows that

$$\boxed{w_1 = y}$$

$$w_1 - w_2 = R_2\left(u_1 - \frac{y}{R_1}\right) \quad \text{or} \quad w_2 = \overset{y}{w_1} - R_2\left(u_1 - \overset{w_1}{\frac{y}{R_1}}\right)$$

$$\boxed{w_2 = \left(1 + \frac{R_2}{R_1}\right) y - R_2 u_1}$$

Lets now write eqs $\boxed{A}$ & $\boxed{B}$ in matrix-vector form.

$\boxed{\text{Example 11}}$ (Example 10 via Nodal Analysis)

Here we write eqs $\boxed{A}$ & $\boxed{B}$ in matrix-vector form.

$$\boxed{A} \Rightarrow \left(\frac{1}{R_1}+\frac{1}{R_2}\right) w_1 + \left(\frac{-1}{R_2}\right) w_2 = u_1$$

$$\boxed{B} \Rightarrow \left(\frac{-1}{R_2}\right) w_1 + \left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right) w_2 = \left(\frac{1}{R_4}\right) u_2$$

From the above, we get the following matrix-vector form:

$$\underbrace{\begin{bmatrix} \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \left(-\frac{1}{R_2}\right) \\ \left(-\frac{1}{R_2}\right) & \left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} u_1 \\ \left(\frac{1}{R_4}\right) u_2 \end{bmatrix}}_{b}$$

The above is in the standard matrix-vector form:

$$Ax = b \, !$$

<u>Note</u>: If you had values for $R_1, R_2, R_3, R_4, u_1, u_2$, then you can feed A & b to a descent calculator so it can solve for w_1, w_2!
(or MATLAB)

Example 11 (using Calculator or MATLAB)

How do we feed (A,b) to your calculator (or MATLAB) to compute w_1, w_2?

Lets rewrite our $Ax=b$ as follows:

$$\underbrace{\begin{bmatrix} \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \left(-\frac{1}{R_2}\right) \\ \left(-\frac{1}{R_2}\right) & \left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{R_4} \end{bmatrix}}_{b_2} u_2 = b$$

Now suppose all $R_i = 1$:

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_2} u_2 = b$$

If (A, b_1) yield x_1 (i.e. $Ax_1 = b_1$) & (A, b_2) yield x_2 (i.e. $Ax_2 = b_2$), then the general solution (by linearity) is given by:

$$x = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} u_1 + \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} u_2$$

Why?

$$\begin{aligned} Ax &= A(x_1u_1 + x_2u_2) \\ &= (Ax_1)u_1 + (Ax_2)u_2 \\ &= b_1u_1 + b_2u_2 \\ &= b \end{aligned}$$

Here's some Material from Linear Algebra: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

Inverse of matrix A

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$$

Note: $$A\,A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc} = \frac{\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{bmatrix}}{ad-bc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For our $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$, $A^{-1} = \frac{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}{6-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$$Ax_1 = b_1 \implies x_1 = A^{-1}b_1 = \begin{bmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 1/5 \end{bmatrix}$$

$$Ax_2 = b_2 \implies x_2 = A^{-1}b_2 = \begin{bmatrix} " & " \\ " & " \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

Can be obtained from calculator or MATLAB!

$$\implies \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}u_1 + \frac{1}{5}u_2 \\ \frac{1}{5}u_1 + \frac{2}{5}u_2 \end{bmatrix}$$

$$x = \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} u_1 + \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} u_2$$

Agrees with w_i on page 291

Problem 11 (Propagation & Nodal Analysis)

(a) Solve Example 10 for y when $R_1 = R_2 = R_3 = R_4 = 1$

(b) Solve Example 11 for W_1, W_2 when $R_1 = R_2 = R_3 = R_4 = 1$

Show that $W_1 = y$

$$W_2 = \left(1 + \frac{R_2}{R_1}\right) y - R_2 u_1$$

i.e. What we get in (b) via nodal analysis MUST agree with what we get in (a) !!!
↑ via propagation

(c) Redo Problem 10 using nodal voltages W_1, W_2 (with $R_i = 1$).

You need to show that $W_1 = V_1 = y$

$$W_2 = V_3 = V_1 - V_2$$

i.e what we get via nodal analysis MUST agree with what we got in Problem 10 !!!

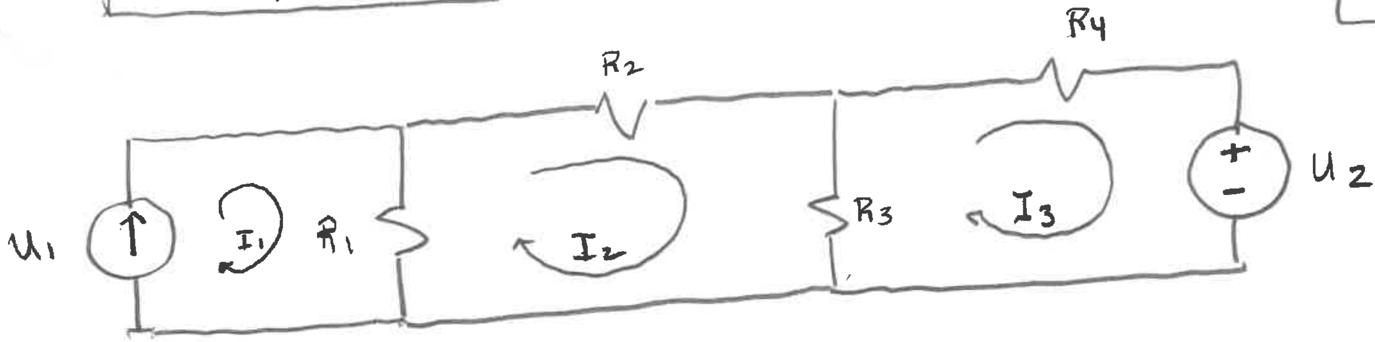
Important Observations:

1. In Example 10, we show that the variable y is all that we need to analyze the circuit.
One y is found, any voltage or current in the circuit may be found in terms of our expression for y !!!

2. In Example 11, we use two variables w_1, w_2 to analyze the circuit. Here, the resulting steps are simpler than in Example 10 but we MUST solve 2 eqs in the 2 unknowns w_1, w_2 .... the resulting simplicity comes at a price...

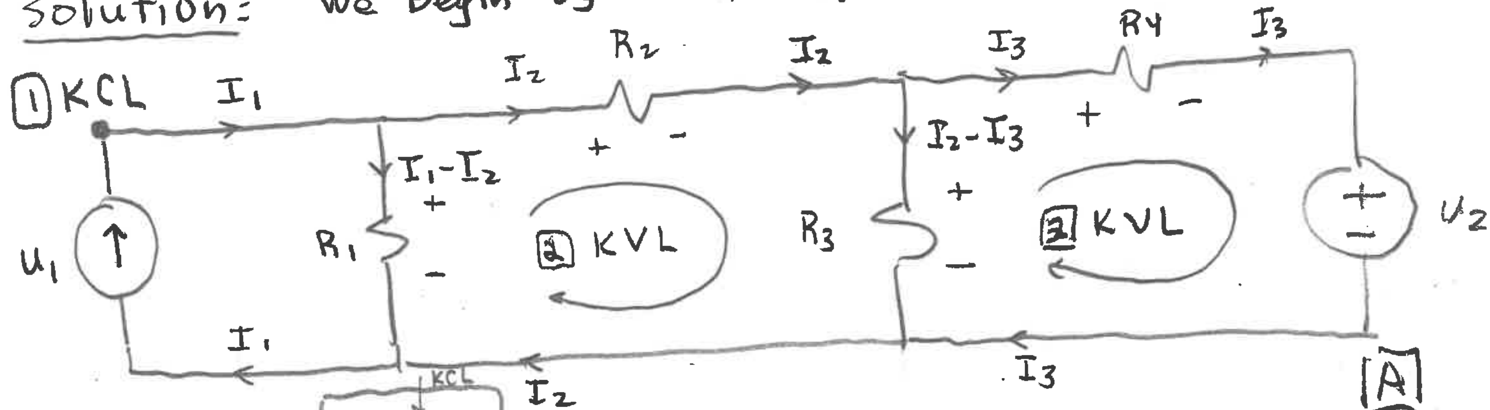
... NOTHING IS EVER FREE!

Example 12 (Example 10 via Mesh Analysis)



Relate the mesh currents I_1, I_2, I_3 to u_1, u_2 ($\ni R_1, R_2, R_3, R_4$)

Solution: We begin by interpreting I_1, I_2, I_3 on the circuit:



1 KCL =	$\boxed{I_1 = u_1}$	A
2 KVL = (+ohm)	$\boxed{R_1(I_1 - I_2) = R_2 I_2 + R_3(I_2 - I_3)}$	B
3 KVL = (+ohm)	$\boxed{R_3(I_2 - I_3) = R_4 I_3 + u_2}$	C

Annotations on B and C: ohm, KVL, ohm, ohm.

$$I_3 = \frac{R_3 I_2 - u_2}{R_3 + R_4}$$

This gives us 3 eqs in the 3 unknowns I_1, I_2, I_3! Solution for I_1, I_2, I_3 must agree with our results from Example 10 / or Example 11.

Note: From Example 10 (pg 270), it follows that

$$R_1(I_1 - I_2) = y \implies I_2 = \underbrace{I_1}_{I_1 = u_1} - \frac{y}{R_1} = u_1 - \frac{y}{R_1}$$

From Example 11 (pg 310), $R_1(I_1 - I_2) = w_1$ & $R_3(I_2 - I_3) = w_2$ ... etc...

Lets now write eqs A, B, C in matrix-vector form.

Example 12

(Example 10 via Mesh Analysis)

Here, we write eqs $\boxed{A}$, $\boxed{B}$, $\boxed{C}$ in matrix-vector form.

$\boxed{A} \Rightarrow \quad I_1 = u_1$

$\boxed{B} \Rightarrow \quad -R_1 I_1 + (R_1 + R_2 + R_3) I_2 - R_3 I_3 = 0$

$\boxed{C} \Rightarrow \quad -R_3 I_2 + (R_3 + R_4) I_3 = -u_2$

From the above, we get the following matrix-vector form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -R_1 & (R_1 + R_2 + R_3) & -R_3 \\ 0 & -R_3 & (R_3 + R_4) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} u_1 \\ 0 \\ -u_2 \end{bmatrix}}_{b}$$

The above is in the standard matrix-vector form:

$$Ax = b \, !$$

Note: If you had values for $R_1, R_2, R_3, R_4, u_1, u_2$, then you can feed A & b to a descent calculator (or MATLAB) so it can solve for I_1, I_2, I_3!

Example 12 (using calculator or MATLAB)

How do we feed (A, b) to your calculator or MATLAB to compute $I_{1,2,3}$?

Lets rewrite our $Ax=b$ as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -R_1 & (R_1+R_2+R_3) & -R_3 \\ 0 & -R_3 & R_3+R_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{b_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{b_2} u_2 = b$$

Now suppose all $R_i = 1$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{b_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{b_2} u_2 = b$$

If (A, b_1) yields x_1 (i.e. $Ax_1 = b_1$) & (A, b_2) yields x_2 (i.e. $Ax_2 = b_2$), then the general solution (by linearity) is given by:

$$x = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} u_1 + \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} u_2$$

Why?

$$\begin{aligned} Ax &= A(x_1u_1 + x_2u_2) \\ &= (Ax_1)u_1 + (Ax_2)u_2 \\ &= b_1u_1 + b_2u_2 \\ &= b \end{aligned}$$

From above eqs, it follows that $\boxed{I_1 = u_1}$

$$-I_1 + 3I_2 - I_3 = 0 \Rightarrow 3I_2 - I_3 = I_1 = u_1$$

$$-I_2 + 2I_3 = -u_2 \Rightarrow -3I_2 + 6I_3 = -3u_2$$

$$5I_3 = u_1 - 3u_2$$

$$\boxed{I_3 = \left(\tfrac{1}{5}\right)u_1 - \left(\tfrac{3}{5}\right)u_2}$$

$$\Rightarrow I_2 = 2I_3 + u_2 = \left(\tfrac{2}{5}\right)u_1 - \left(\tfrac{6}{5}\right)u_2 + u_2$$

$$\Rightarrow \boxed{I_2 = \left(\tfrac{2}{5}\right)u_1 - \left(\tfrac{1}{5}\right)u_2}$$

Note:
The simplicity of Mesh (& Nodal) analysis comes at a price --- More equations that must be solved simultaneously!

Note: $I_{1,2,3}$ here agree with that found on page 291!

Problem 12 (Propagation, Nodal, & Mesh Analysis)

(a) Solve Example 10 for y when $R_1 = R_2 = R_3 = R_4 = 1$.

(b) Solve Example 11 for w_1, w_2 " " " .

(c) Solve Example 12 for I_1, I_2, I_3 " " " .

(d) Show that the answers obtained in (a), (b), (c) are in agreement!

i.e.

Show w_1, w_2 from Example 11 satisfy:

$$w_1 = y$$
$$w_2 = \left(1 + \frac{R_2}{R_1}\right) y - R_2 u_1$$

Show I_1, I_2, I_3 from Example 12 satisfy:

$$I_1 = u_1$$
$$I_2 = u_1 - \frac{y}{R_1}$$
$$I_3 = \frac{R_3 I_2 - v_2}{R_3 + R_4}$$

Note: The simplicity of Nodal & Mesh analysis comes at a price... — MORE EQUATIONS THAT MUST BE SOLVED SIMULTANEOUSLY!

(e) Repeat the above for Problem 10.

i.e. show that you get same results

via propagation (Problem 10)
nodal analysis (Problem 11C)
mesh analysis

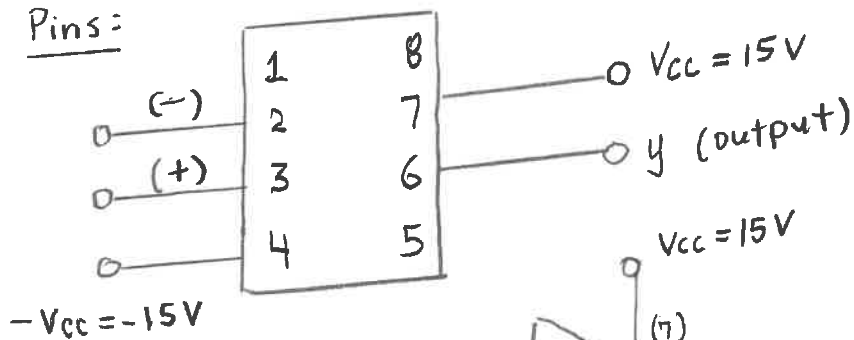
Example 13

(Introduction to 741 Operational Amplifier, Ideal/Standard Op-Amp Assumptions, Buffer/Follower Amplifier, & Its use!)

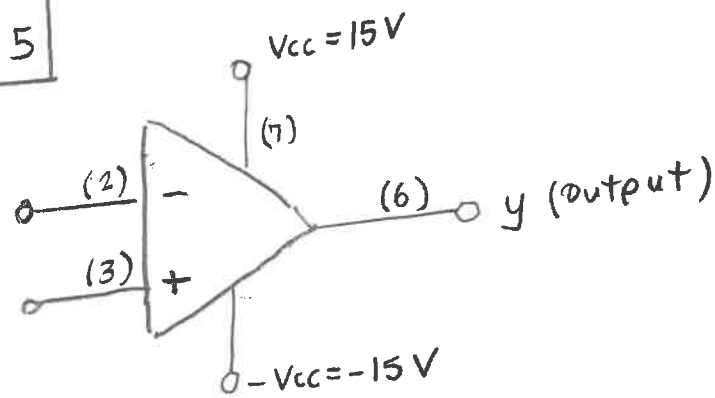
741 Operational Amplifier

(David Fullagar, Fairchild Semiconductor 1968)

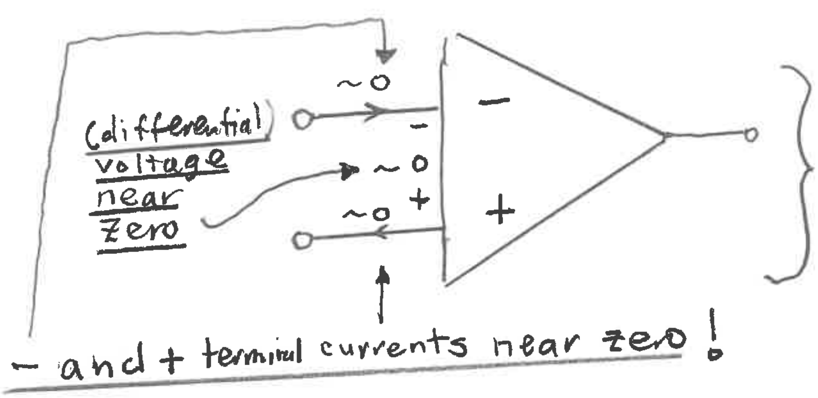
Pins:



Schematic:



Ideal Op-Amp Assumptions



Note:

These ideal op-amp assumptions are a consequence of the following standard (limiting) op-amp assumptions:

(input resistance) R_{in} large $\rightarrow \infty$

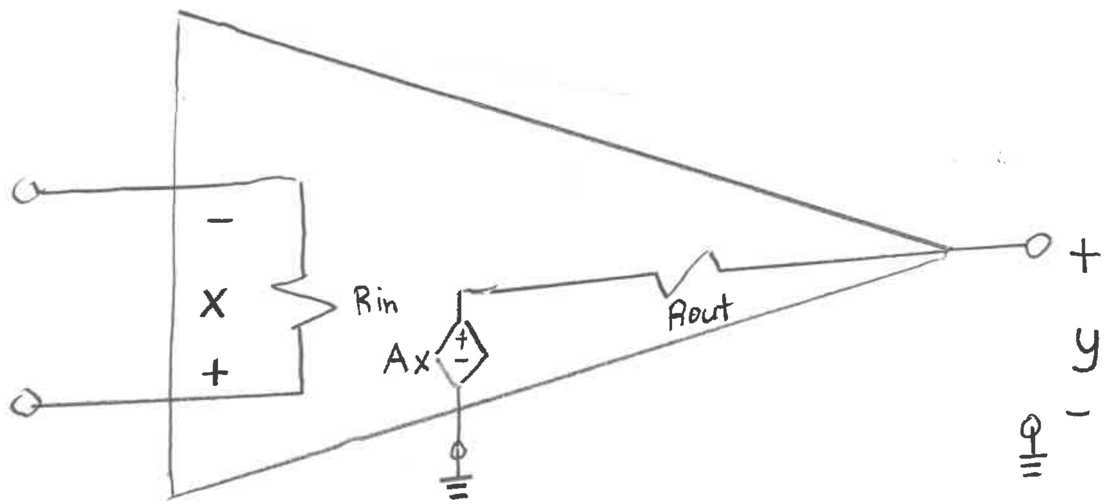
(output resistance) R_{out} small $\rightarrow 0$

(gain) A large $\rightarrow \infty$

lets explain a little

Example 13

Standard Op-Amp Model



Rin – input resistance

Rout – output resistance

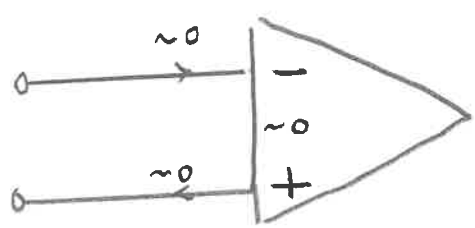
Ax – voltage dependent voltage source

As $R_{in} \to \infty$ (Infinite input resistance)

$R_{out} \to 0$ (Zero output resistance)

$A \to \infty$ (Infinite gain)

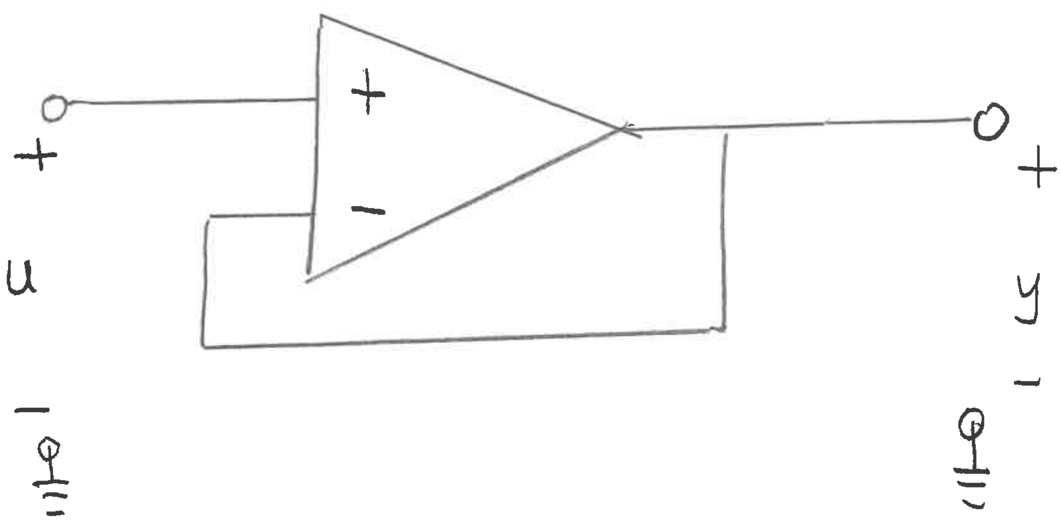
we get the ideal op-amp assumptions:



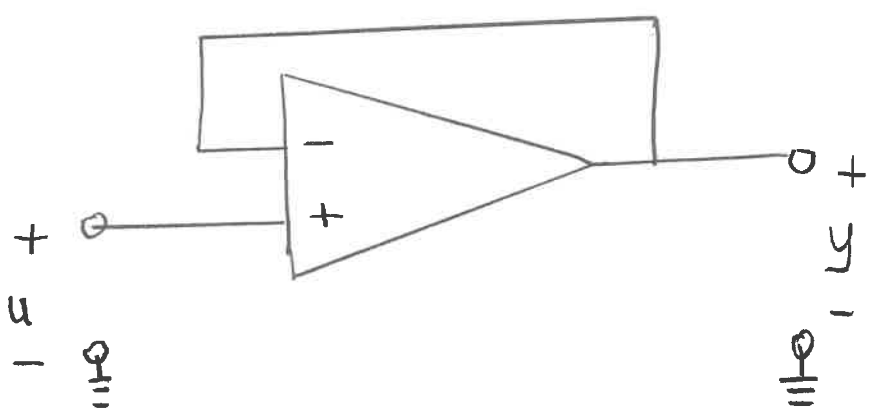
Now, lets examine the Buffer/Follower Amplifier ...

Example 13

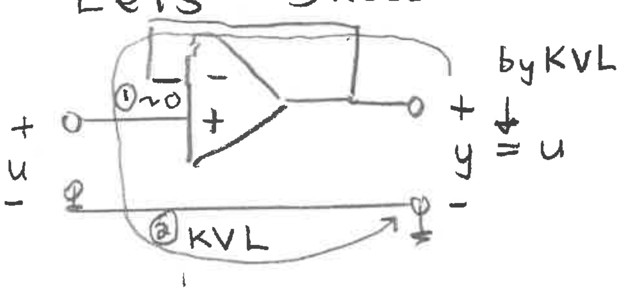
Buffer/Follower Amplifier



or



Lets show that → $\boxed{y = u}$

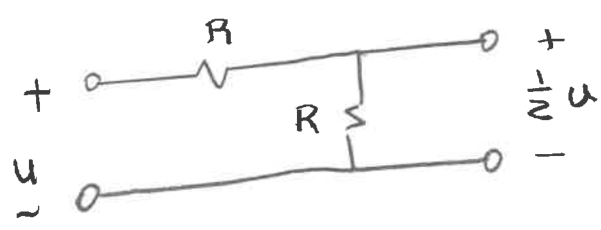


Buffer/Follower
(under ideal op-amp assumptions)
... How/Why is this useful?

Example 13

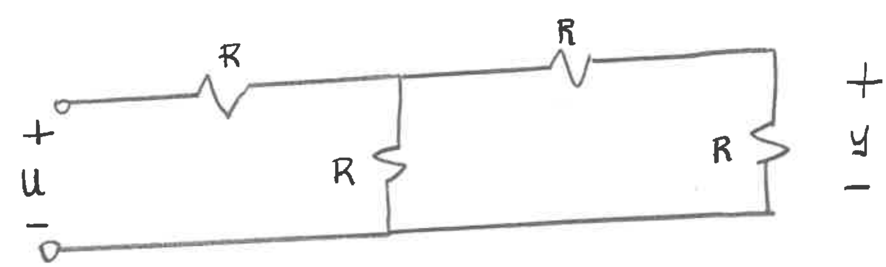
Why is a Buffer/Follower Amplifier useful?

Consider the simple voltage divider circuit:



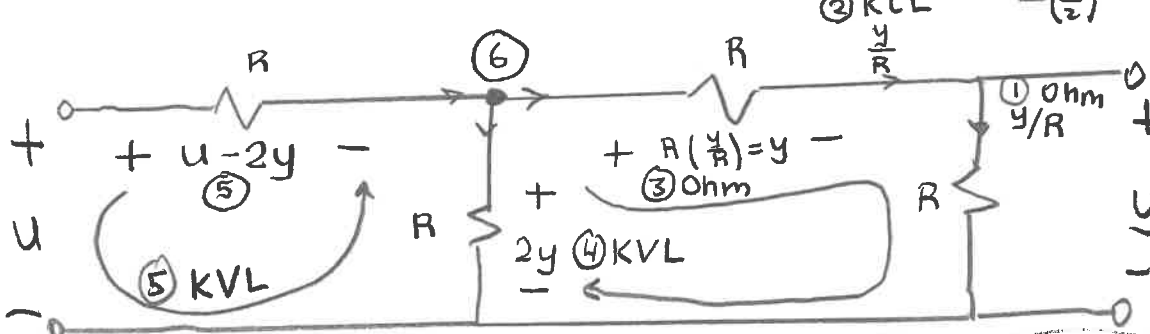
$$\frac{1}{2}u = \left(\frac{R_2}{R_1+R_2}\right)u\Big|_{R_i=R} = \left(\frac{R}{2R}\right)u = \frac{1}{2}u$$

Now consider 2 of these cascaded:



Lets relate y to u ... (using KVL, KCL, Ohm)

Note: it's not going to be $\frac{1}{4}u$!!! ← $\left(\frac{1}{2}\right)^2$



⑥ KCL (+ohm) =

$$\boxed{\underbrace{\left(\frac{u-2y}{R}\right)}_{\text{ohm}} \underset{\text{KCL}}{=} \left(\frac{2y}{R}\right) + \left(\frac{y}{R}\right)}$$

ohm ohm

Now we do algebra ...

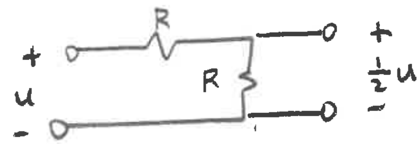
Example 13

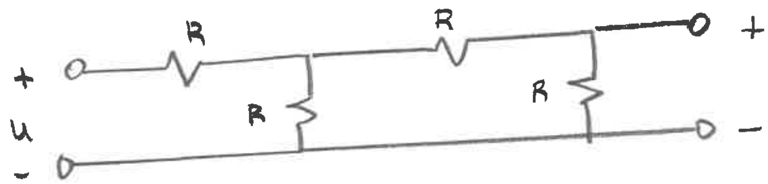
Algebra = $y\left[\underbrace{\frac{2}{R}+\frac{2}{R}+\frac{1}{R}}_{\frac{5}{R}}\right]=\left[\frac{1}{R}\right]u$

$\overset{\text{or}}{\Longrightarrow}\quad y=\frac{1}{5}u \neq \frac{1}{4}u$

Summary:

Even though

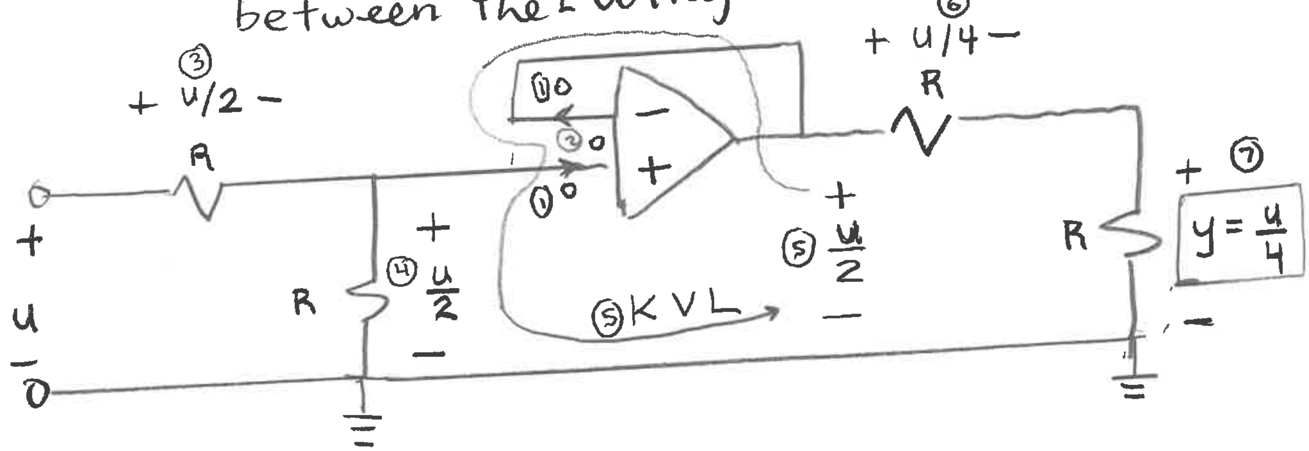




$\frac{1}{5}u \neq \frac{1}{4}u$

How do we get the desired $y=\frac{1}{4}u$?

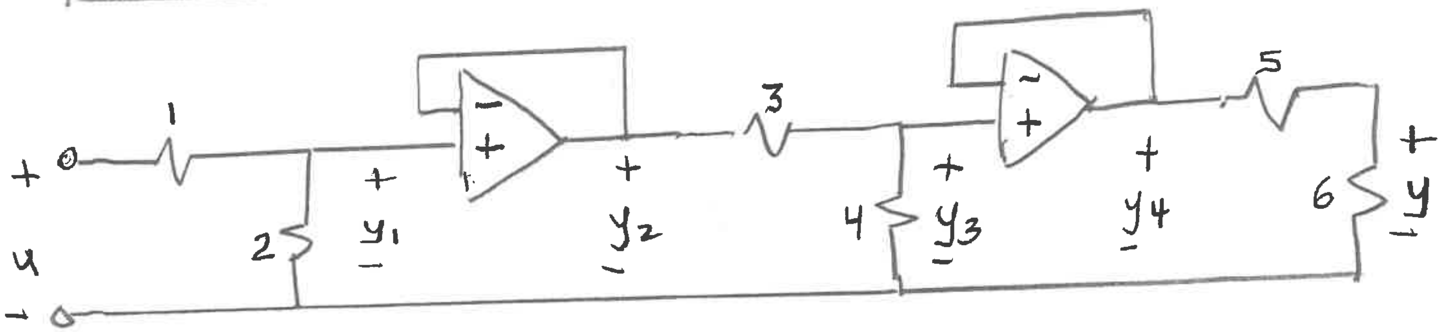
Answer: Use a Buffer/Follower Amplifier between the 2 voltage dividers:



Buffer/Follower Amplifier permits us to design circuits systematically & modularly ... to get the desired "cascaded product" result by inserting buffer/follower in between

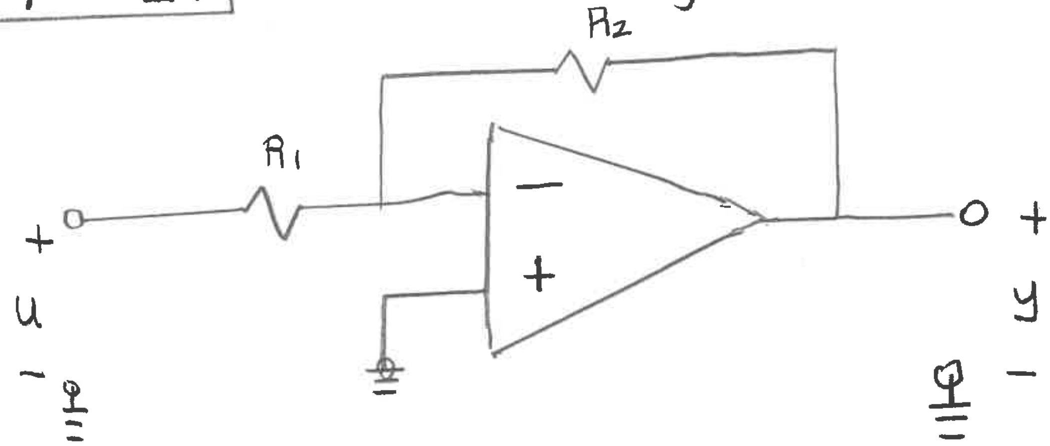
Problem 13

420



Relate $\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y \end{Bmatrix}$ to u.

Example 14 (Inverting Amplifier)

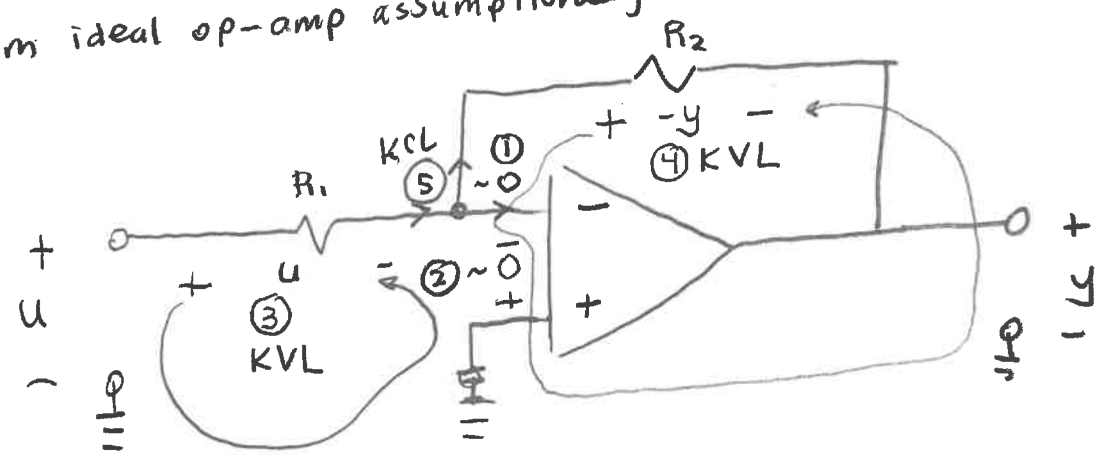


a Show that $$\boxed{y = -\left(\frac{R_2}{R_1}\right) u}$$

(using ideal op-amp assumptions)

<u>solution:</u>

From ideal op-amp assumptions, we have:



⑤ KCL (& ohm):

KCL

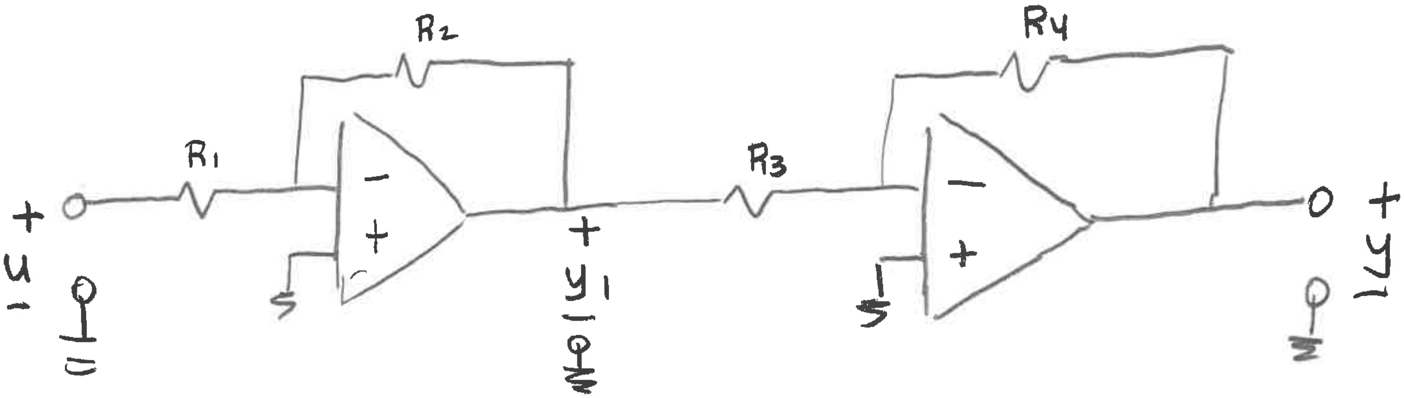
$$\boxed{\underbrace{\left(\frac{u}{R_1}\right)}_{\text{ohm}} = \underbrace{\left(\frac{-y}{R_2}\right)}_{\text{ohm}} + 0}$$

$\xRightarrow{\text{algebra}}$ $\boxed{y = -\left(\frac{R_2}{R_1}\right) u}$

-- the desired result!

Example 14

(b) Now consider 2 cascaded inverting amplifiers:



From ideal op-amp assumptions, it follows that

$$y_1 = -\left(\frac{R_2}{R_1}\right) u$$

&

$$y = -\left(\frac{R_4}{R_3}\right) y_1$$

or

$$\boxed{y = \left(\frac{R_4}{R_3}\right)\left(\frac{R_2}{R_1}\right) u}$$

Note: Suppose we desire $y = g\,u$.
To achieve this, we can use $R_2 = g\,R_1$
& $R_4 = R_3$

Here, we can achieve any gain g
(>1, $=1$, or <1 !!!)

Problem 14

Show how inverting amplifiers can be used to achieve:

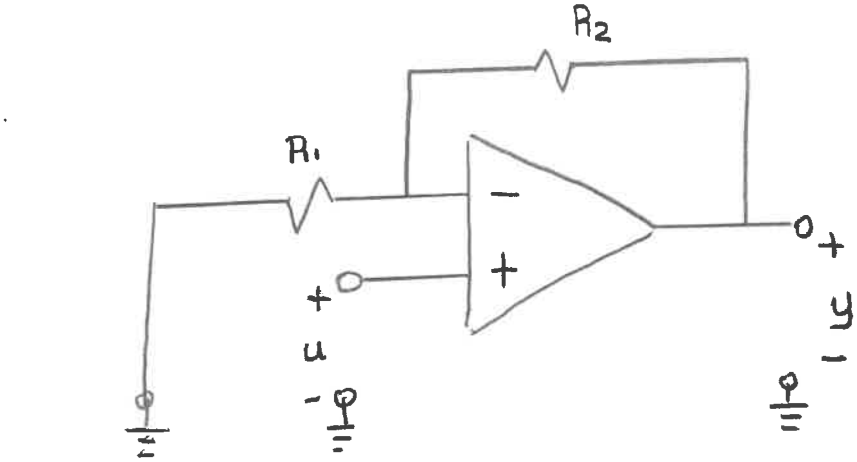
a) $y = -10\,u$

b) $y = 10\,u$

c) $y = \frac{1}{5}\,u$

<u>Note</u>: The above is "design!"

Example 15 (Non-Inverting Amplifier)

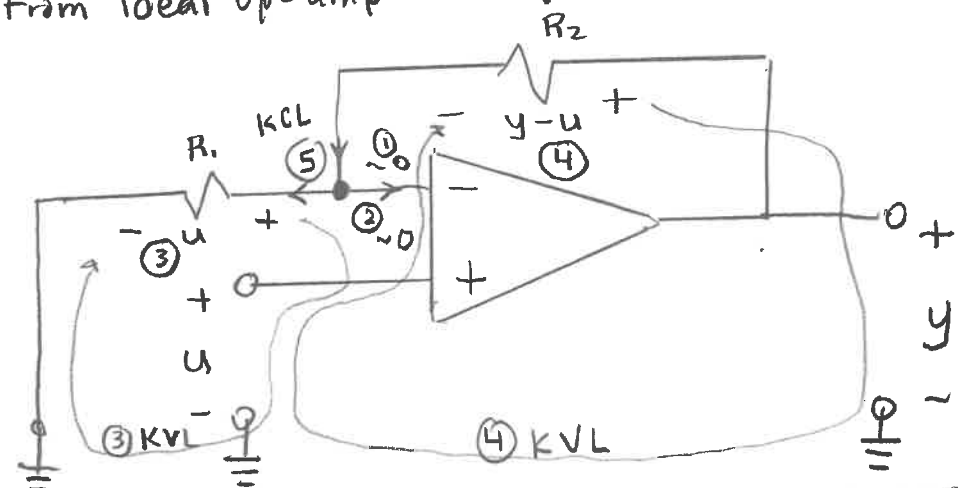


[a] Show that $\boxed{y = \left(1 + \frac{R_2}{R_1}\right) u}$

(using ideal op-amp assumptions)

solution:

From ideal op-amp assumptions, we have:



5 KCL (+Ohm) = $\boxed{\overset{\text{ohm}}{\left(\frac{u}{R_1}\right)} + 0 \overset{\text{KCL}}{=} \overset{\text{ohm}}{\left(\frac{y-u}{R_2}\right)}}$

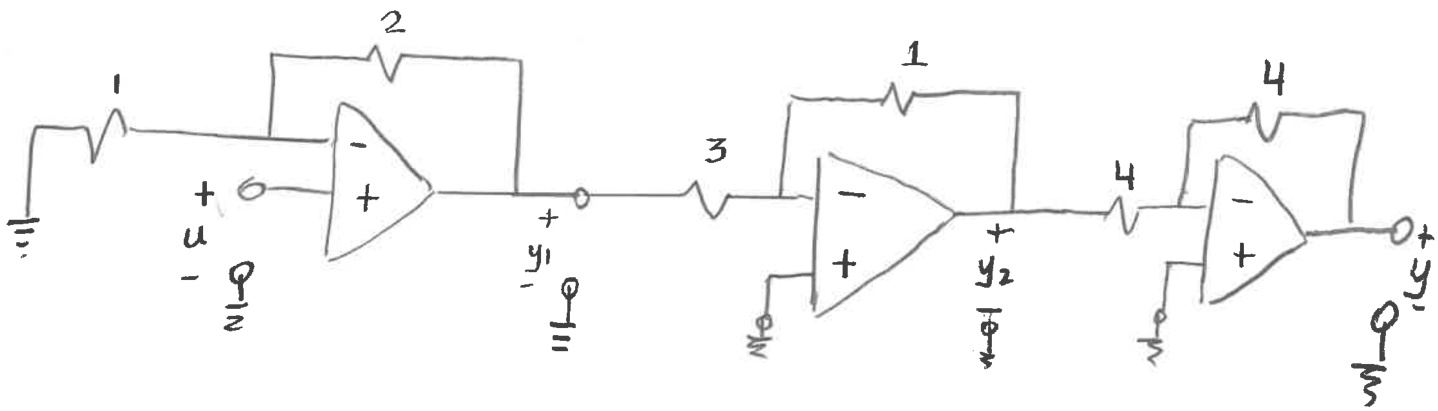
$\xrightarrow{\text{algebra}} \quad \frac{y}{R_2} = u\left[\frac{1}{R_1} + \frac{1}{R_2}\right] \overset{\text{Multiply both sides by } R_2}{\Longrightarrow} \boxed{y = \left[1 + \frac{R_2}{R_1}\right] u}$

--- the desired result !

Note: To achieve $y = 10u$, we just need $R_2 = 9R_1$.

To achieve $y = \frac{1}{2}u$, we need $R_2 = -\frac{1}{2}R_1$ (Negative Resistance !!! ... more on this later ...)

Problem 15



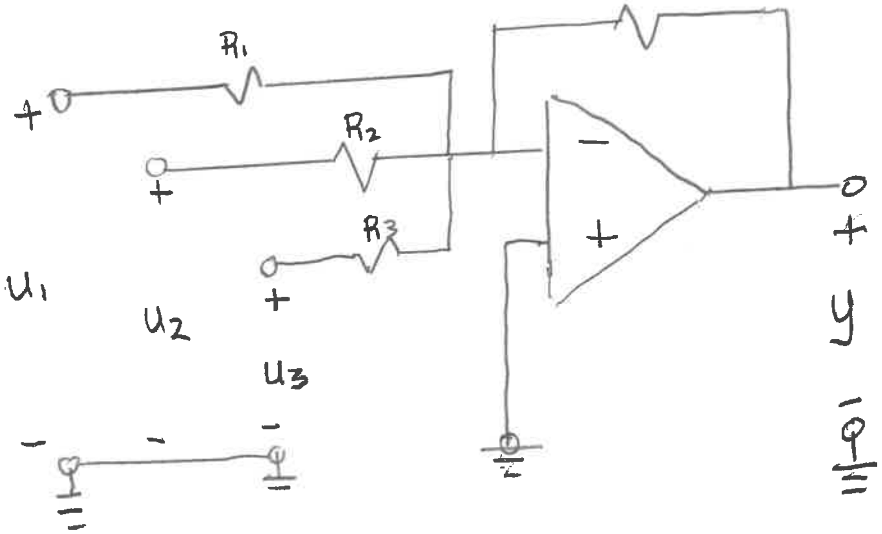
Relate a) y_1 to u

b) y_2 to u

c) y to u

Example 16

(Inverting Summing Amplifier)

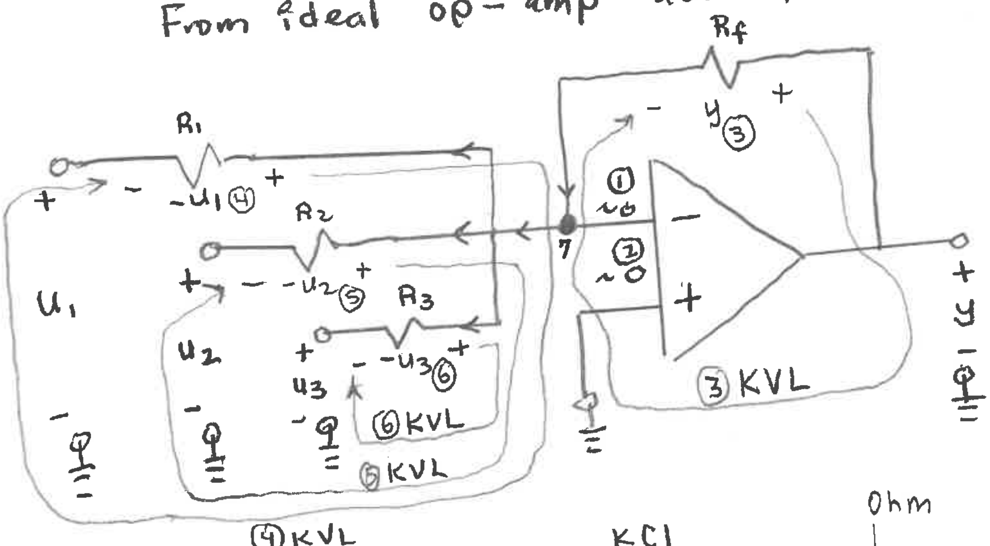


a Show that $\boxed{y = -\left(\frac{R_f}{R_1}\right)u_1 - \left(\frac{R_f}{R_2}\right)u_2 - \left(\frac{R_f}{R_3}\right)u_3}$

(using ideal op-amp assumptions)

solution:

From ideal op-amp assumptions, we have:



7 KCL (+ Ohm) = $\left(\frac{y}{R_f}\right) = \left(\frac{-u_1}{R_1}\right) + \left(\frac{-u_2}{R_2}\right) + \left(\frac{-u_3}{R_3}\right)$

(ohm, KCL, Ohm, Ohm, Ohm)

~0 $\Rightarrow$ $y = -\left(\frac{R_f}{R_1}\right)u_1 - \left(\frac{R_f}{R_2}\right)u_2 - \left(\frac{R_f}{R_3}\right)u_3$

... the desired result!

Problem 16

Show how to achieve

$$y = g\left[\left(\frac{R_f}{R_1}\right)u_1 + \left(\frac{R_f}{R_2}\right)u_2 + \left(\frac{R_f}{R_3}\right)u_3\right]$$

<u>Hint:</u> place inverter with gain $\left(-\frac{R_2}{R_1}\right) = -g$

on output of circuit in Example 16.

Show your final circuit in your solution!

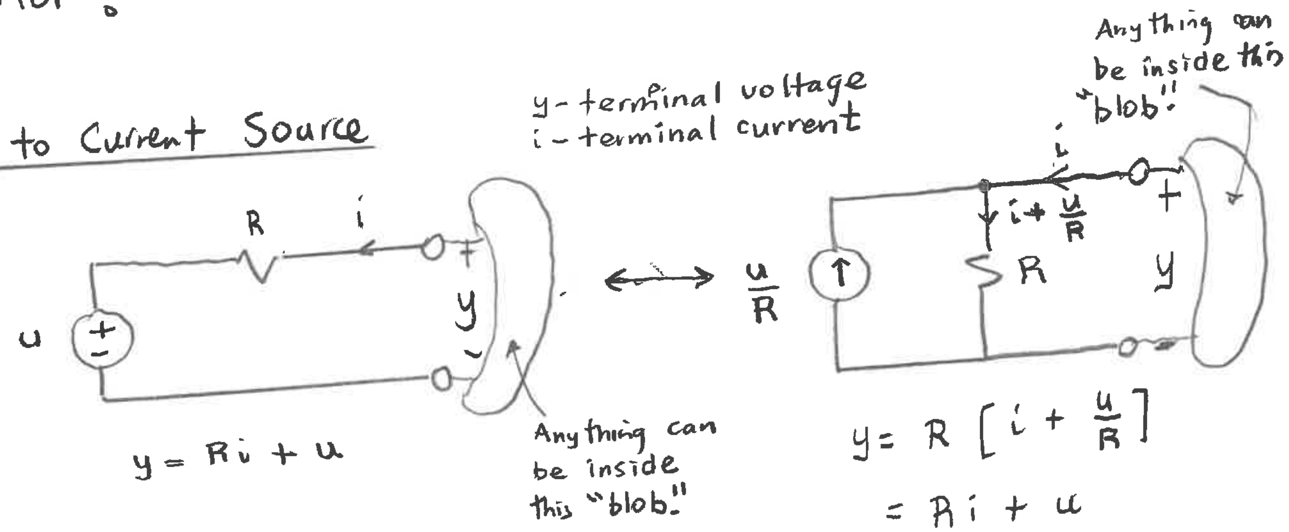
Overview of Source Transformation Concepts

Questions:

1. How can we transform a voltage source in series with a resistor to a current source in parallel with a resistor?
2. How can we transform a current source in parallel with a resistor to a voltage source in series with a resistor?

Voltage to Current Source

y - terminal voltage
i - terminal current



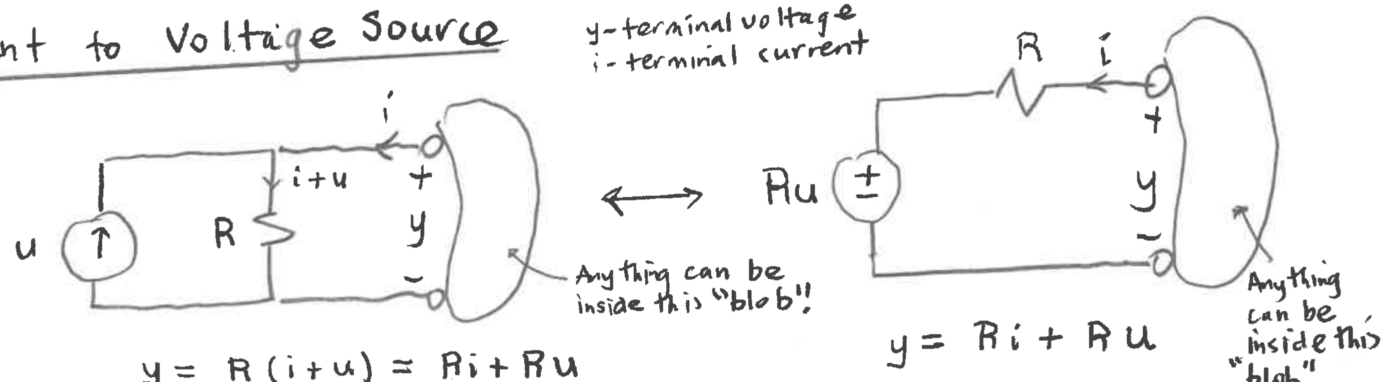
$y = Ri + u$

$y = R\left[i + \frac{u}{R}\right]$
$= Ri + u$

The above 2 circuits are said to be equivalent because they have the exact same y-i relationship between the terminal voltage y & the terminal current i.

Current to Voltage Source

y - terminal voltage
i - terminal current



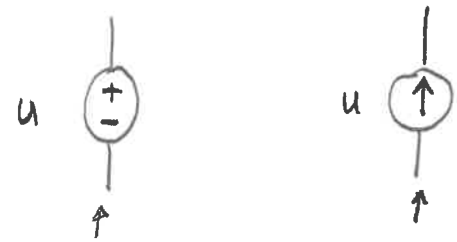
$y = R(i+u) = Ri + Ru$

$y = Ri + Ru$

The above 2 circuits are also said to be equivalent from a terminal y-i perspective!!!

A Practical Note on Real Voltage & Current Sources

These are ideal sources! →



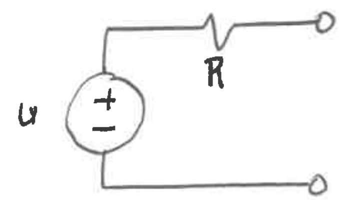
Ideal Voltage Source

Ideal Current Source

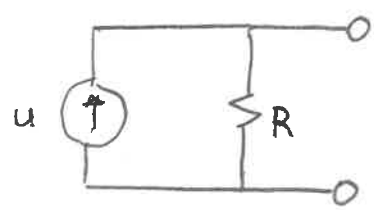
<u>Note</u>: These only exist in books ... NOT in the real world!

In the real world

(1) <u>A Real Voltage Source</u> must have a <u>series resistance</u>:



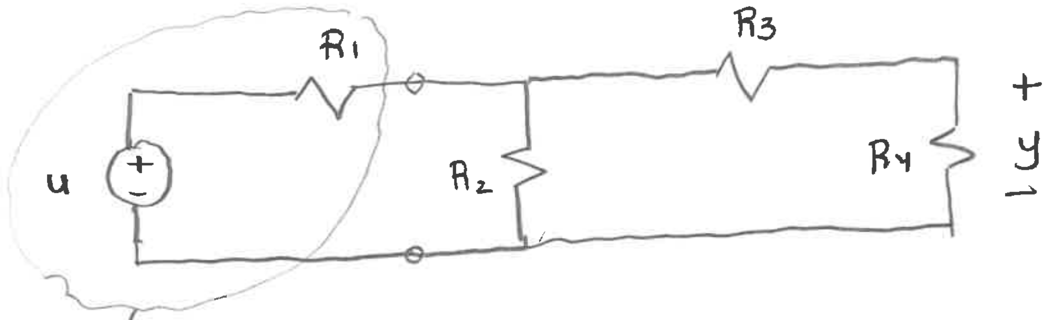
(2) <u>A Real Current Source</u> must have a <u>parallel resistance</u>:



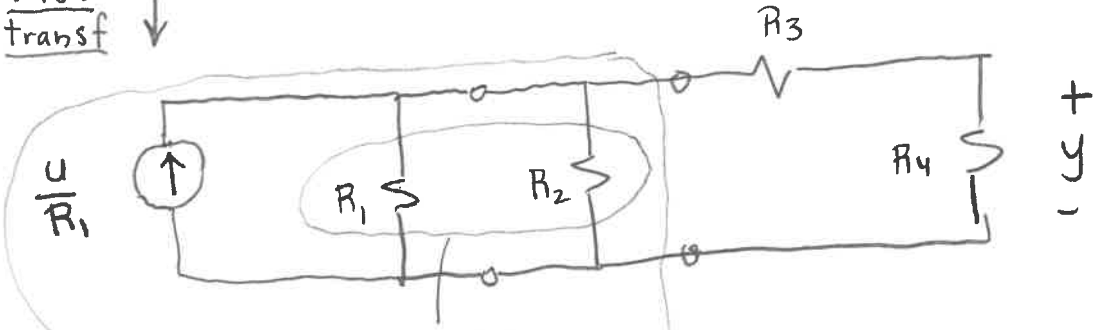
Example 17 (Example 7a via Source Transformations)

510

Relate y to u (via source transformations)



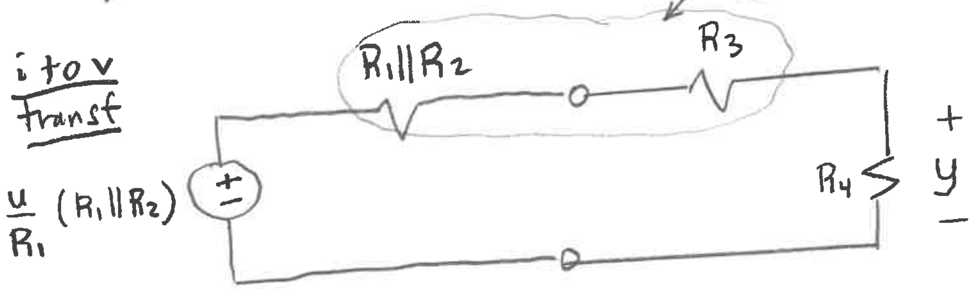
① v to i transf



② $R_1 \| R_2$ (R_1 & R_2 in parallel)

④ $(R_1\|R_2)+R_3$ ($(R_1\|R_2)$ is in series with R_3)

③ i to v transf



⑤ By voltage division (Example 2, pg 170), we have

$$\boxed{y = \left[\frac{R_4}{[(R_1\|R_2)+R_3] + R_4}\right] \frac{u}{R_1}(R_1\|R_2)}$$

Note: Always a good idea to do a simple check using nice R_i values!

Check for $R_i = 1$ (or R)

$$y = \left[\frac{1}{[(1\|1)+1]+1}\right] \frac{u}{1}(1\|1) = \left[\frac{1}{\frac{1}{2}+1+1}\right] u\left(\tfrac{1}{2}\right) = \left[\frac{1}{\frac{5}{2}}\right]\frac{u}{2} = \frac{u}{5}$$

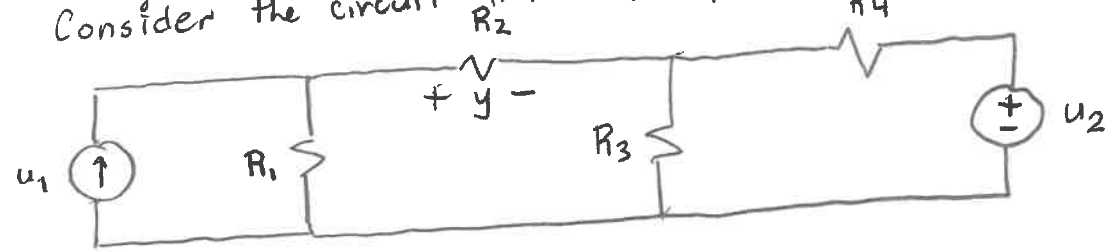
$\frac{(1)(1)}{1+1} = \frac{1}{2}$

$2\frac{1}{2} = \frac{5}{2}$

... as obtained in Example 7a page 170

Problem 17 (Made for Source Transformations!!)

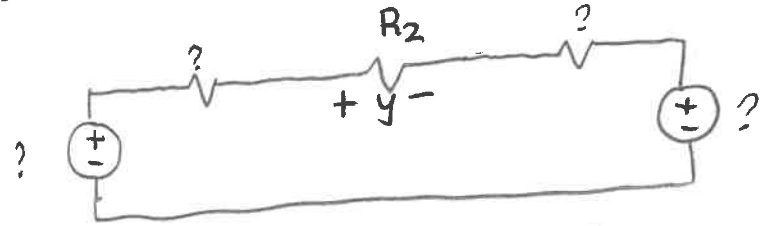
Consider the circuit from Example 10 (page 270)



Hint: To get started apply KCL to get downward current through R_1.

a) Relate y to u (via KVL, KCL, Ohm)

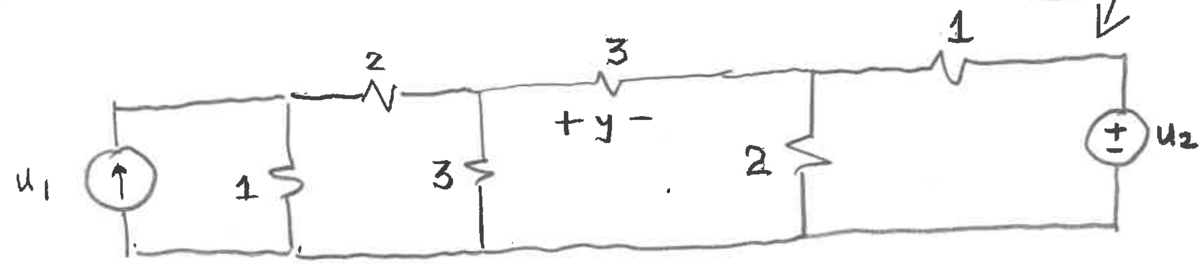
b) Now use source transformations on the left & on the right to get



Then solve for y! (Is source transformation a COOL Method ?)

Our 1st Cool ("Needy") Circuit

Now Consider the more complicated circuit

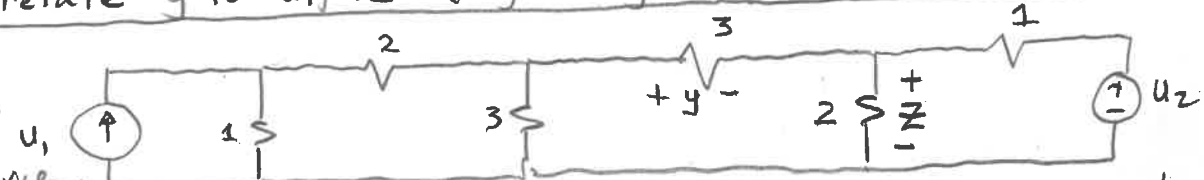


c) Use source transformations to relate y to u_1, u_2

Can this be solved with standard KVL, KCL, Ohm, without introducing another variable? NO!

d) Now relate y to u_1, u_2 (by using Z, KVL, KCL, Ohm)

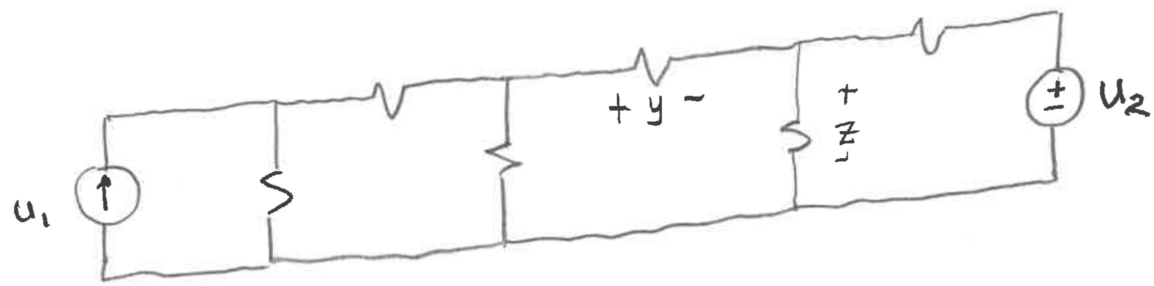
* "Needed" to Introduce Another Variable if we don't use source transf



You'll get 2 eqs in the 2 unknowns y, Z --- solve for y!

Problem 17

Now consider the simplified (All $R_i = 1$) circuit:



[e] Use KVL, KCL, Ohm to relate y, z to u_1, u_2.
You'll get 2 eqs in the 2 unknowns y, z ... solve for y!

[f] Now use source transformations to find y (do NOT use z!!! It is NOT needed when you use source transformations!)

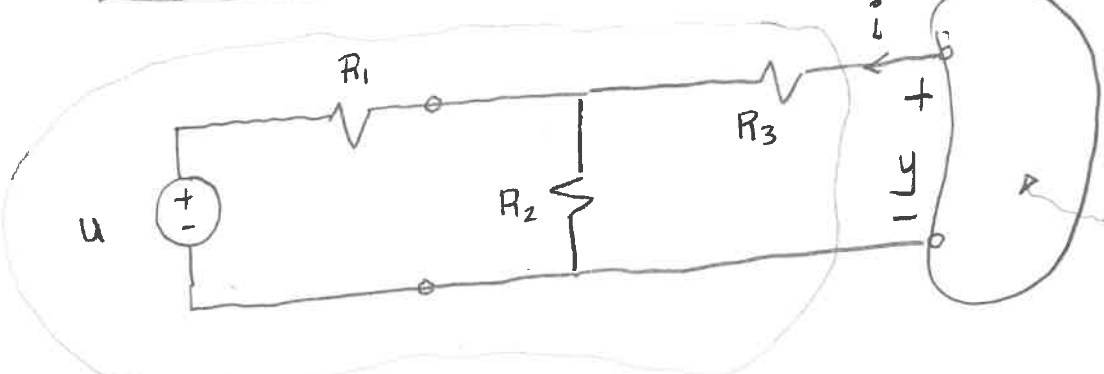
Important Observation / Lesson:

Many circuits are one (1) variable circuits; i.e. the circuit can be completely analyzed using only one (1) variable ... additional variables (such as z above or node voltages in nodal analysis or mesh currents in mesh analysis may be completely unnecessary!!!)

The circuit above is a one (1) variable circuit when source transformations are used; i.e. the introduction of z becomes unnecessary!

A GOOD RULE: DON'T INTRODUCE ADDITIONAL VARIABLES UNLESS NEEDED!

Example 18 (Thevenin Equivalents) & Norton

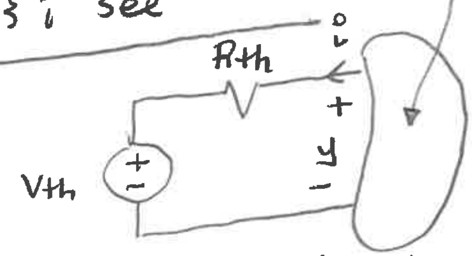


y - terminal voltage

i - terminal current

Anything can be inside this "blob."

Main Question: How can we replace what y & i "see" with the much simpler cct:



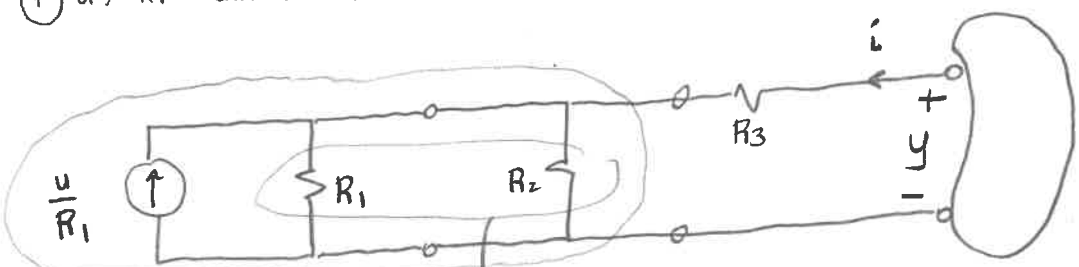
(A "simplification" or "approximation" problem)

$$y = R_{th}\, i + V_{th}$$

Lets answer this in several ways!

a Use source transformations to find V_{th} & R_{th}

① u & R_1 can be transformed to:



② R_1 & R_2 are in parllel ... combine to $R_1 \| R_2 \triangleq \dfrac{R_1 R_2}{R_1+R_2}$

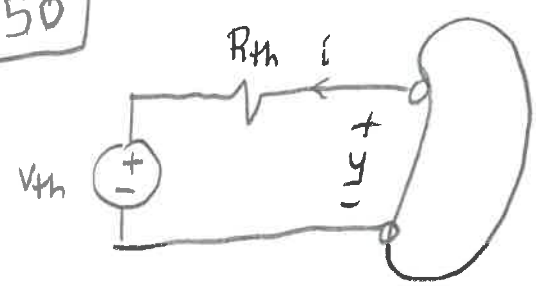
Note: Our analysis has greatly simplified what y sees looking leftward!

⇒ From this, we get

$$V_{th} = \left(\frac{u}{R_1}\right)\left(R_1 \| R_2\right) = \left(\frac{R_2}{R_1+R_2}\right)u \qquad \left(R_1 \| R_2 = \frac{R_1 R_2}{R_1+R_2}\right)$$

& $R_{th} = (R_1 \| R_2) + R_3$

Example 18 (Thevenin & Norton Equivalents) 550



Noting that $y = R_{th}\, i + V_{th}$,

it follows that

when $i = 0$, $y = \underbrace{y_{oc}}_{\text{open circuit}} = V_{th}$

$$\Rightarrow \boxed{V_{th} = y_{oc} \triangleq y\big|_{i=0}}$$

If we set $y = 0$ $(\underbrace{y_{sc}}_{\text{short circuit}} = 0)$, it follows that $0 = R_{th}\, i_{sc} + V_{th}$

$i_{sc} = i\big|_{y=0}$

or

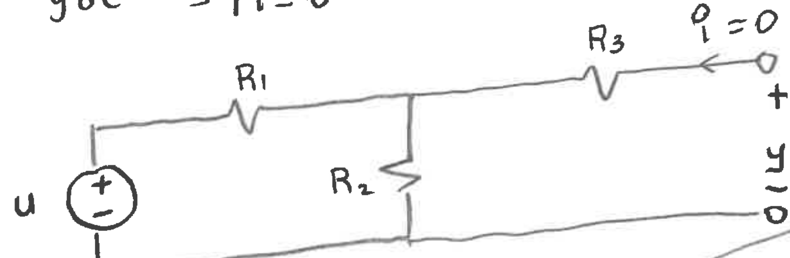
$$\boxed{R_{th} = \frac{-V_{th}}{i_{sc}}}$$

Lets apply the above to our circuit.

b Find $y_{oc} = y\big|_{i=0}$, $V_{th} = y_{oc}$, $i_{sc} = i\big|_{y=0}$, $R_{th} = \frac{-V_{th}}{i_{sc}}$

Find $y_{oc} = y\big|_{i=0}$

This is the relationship between y & u when $i = 0$!



$\frac{R_1 \| R_2}{R_1}$ from page 530

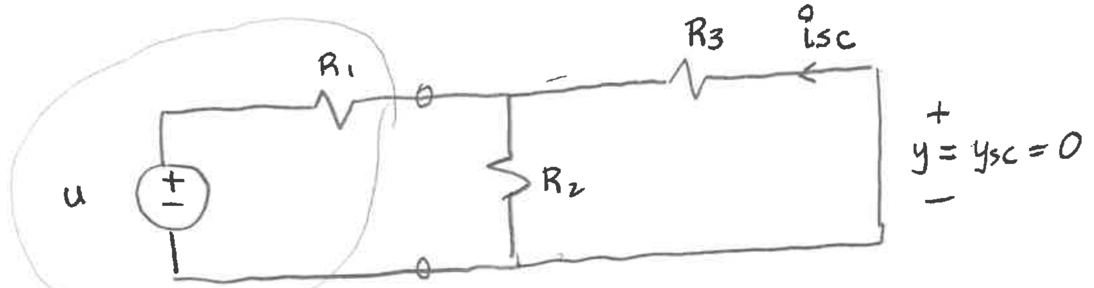
By voltage division

$$V_{th} = y_{oc} = y\big|_{i=0} = \left(\frac{R_2}{R_1 + R_2}\right) u$$

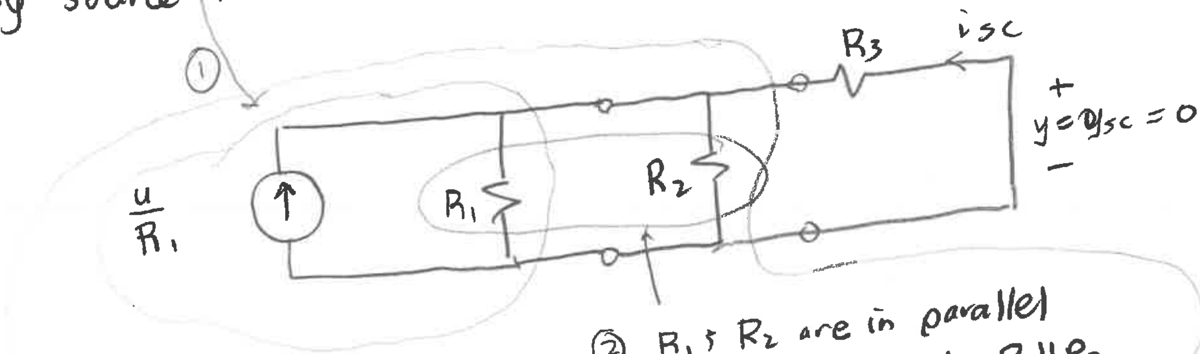
agrees with what we got at bottom of page 530!

Example 18 (Thevenin & Norton Equivalents)

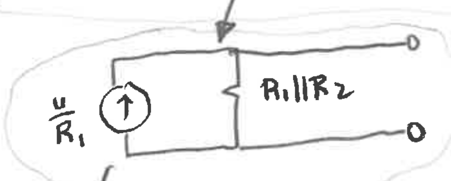
Now Find $i_{sc} = i \big|_{y=0}$

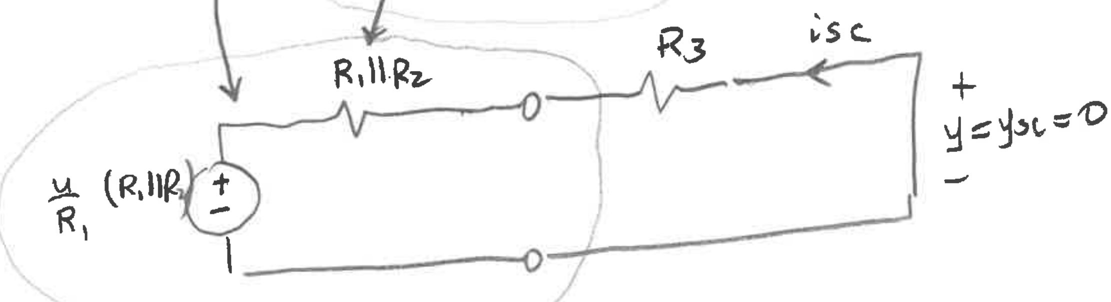


By source transformations:



② R_1 & R_2 are in parallel ... combine to get $R_1 \| R_2$





$$\Rightarrow \quad i_{sc} = \frac{-\frac{u}{R_1}(R_1 \| R_2)}{(R_1 \| R_2) + R_3}$$

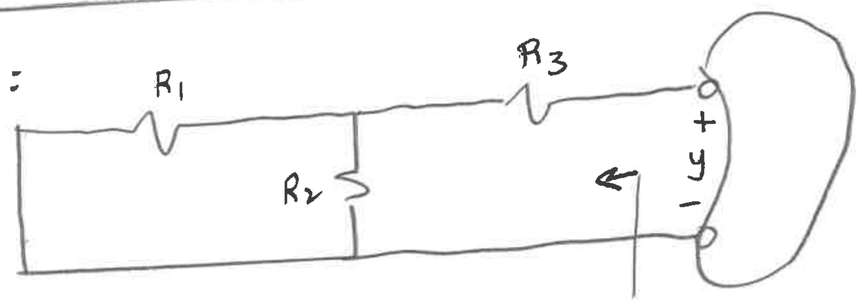
Now, we compute

$$R_{th} = \frac{-V_{th}}{i_{sc}} = \frac{-\frac{(R_1 \| R_2)}{R_1} u}{\frac{-\frac{u}{R_1}(R_1 \| R_2)}{(R_1 \| R_2) + R_3}} = (R_1 \| R_2) + R_3$$

agrees with result on pg 530 !

[c] Set $u=0$ & determine the equivalent resistance as seen from y looking leftward.

solution:



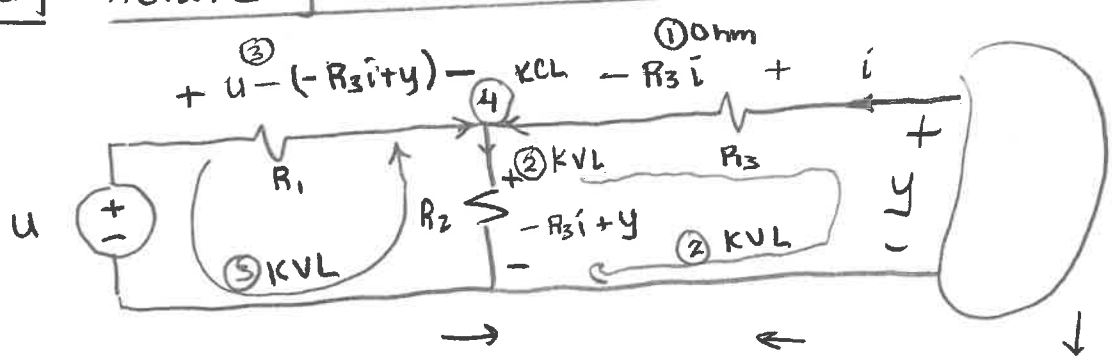
(input voltage u has been set to zero; i.e. shorted)

$$R_{eq} = R_3 + (R_1 \| R_2)$$

... but this is precisely $R_{th} = R_3 + (R_1 \| R_2)$.

THIS IS MOST GENERAL METHOD FOR FINDING A THEVENIN EQUIVALENT

[d] Relate y to i (& u, R_1, R_2, R_3, R_4) via (KVL, KCL, Ohm)



Note: This method is essential when the circuit has dependent sources!

④ KCL = (+Ohm)

$$\left(\frac{u-(-R_3 i+y)}{R_1}\right) + i = \left(\frac{-R_3 i + y}{R_2}\right)$$

$$\frac{u}{R_1} + i\left[\frac{R_3}{R_1} + 1 + \frac{R_3}{R_2}\right] = y\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$\xrightarrow{\text{algebra}}$

$\boxed{\text{Example 18}}$ (Thevenin & Norton Equivalents)

Multiplying both sides by R_1 yields:

$$u + i\left[R_3 + R_1 + \frac{R_1 R_3}{R_2}\right] = y\left[1 + \frac{R_1}{R_2}\right]$$

Multiplying both sides by R_2 yields:

$$R_2 u + i\Big[\underbrace{R_2(R_1+R_3) + R_1 R_3}_{R_1 R_2 + R_3(R_1+R_2)}\Big] = y\left[R_1 + R_2\right]$$

$$y = \Big[\underbrace{\frac{R_1 R_2}{R_1+R_2}}_{R_1 \| R_2} + R_3\Big] i + \Big[\underbrace{\frac{R_2}{R_1+R_2}}_{\frac{R_1 \| R_2}{R_1}}\Big] u$$

$\Rightarrow \quad y = R_{th}\, i + V_{th}$ where

$$R_{th} = (R_1 \| R_2) + R_3$$

$$V_{th} = \left(\frac{R_1 \| R_2}{R_1}\right) u = \left(\frac{R_2}{R_1+R_2}\right) u$$

as we got on page 530

Remember:

Using KVL, KCL, Ohm to relate y to i

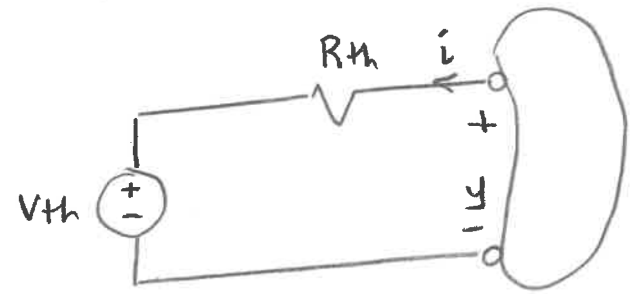
$$y = \underbrace{\boxed{}}_{R_{th}}\, i + \underbrace{\boxed{}}_{V_{th}}$$

is the most general method to find R_{th} & V_{th}.

The other methods are not general; i.e. they do not always work (e.g. when we have dependent sources!!!).

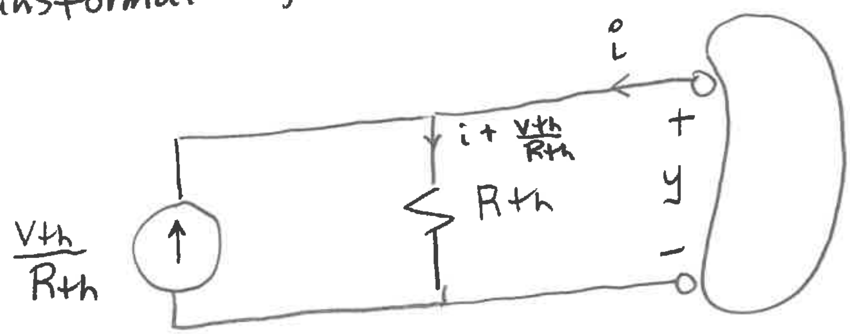
Example 18 (Thevenin & Norton Equivalents)

If our $\boxed{\text{Thevenin equivalent}}$ is



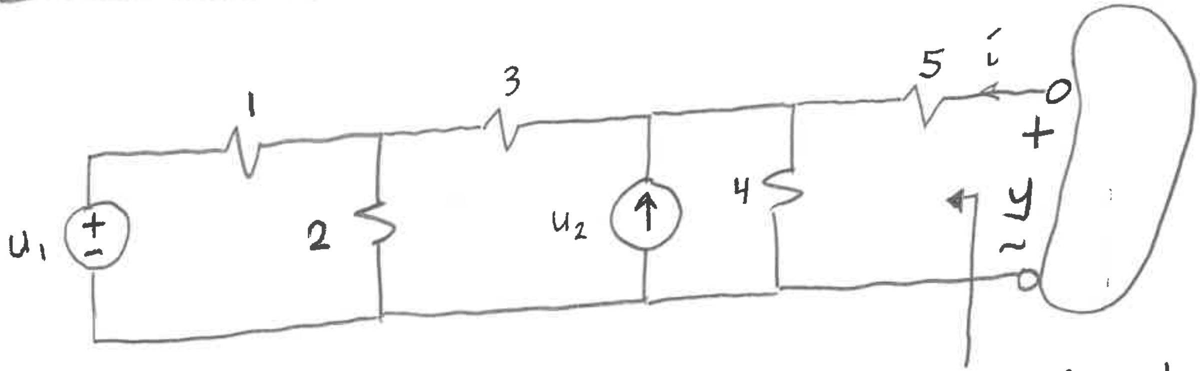
$$y = R_{th} i + V_{th}$$

then the so-called $\boxed{\text{Norton equivalent}}$ (by source transformation) is



$$y = R_{th}\left(i + \frac{V_{th}}{R_{th}}\right)$$
$$= R_{th} i + V_{th}$$

Problem 18 (Thevenin & Norton Equivalent) 600



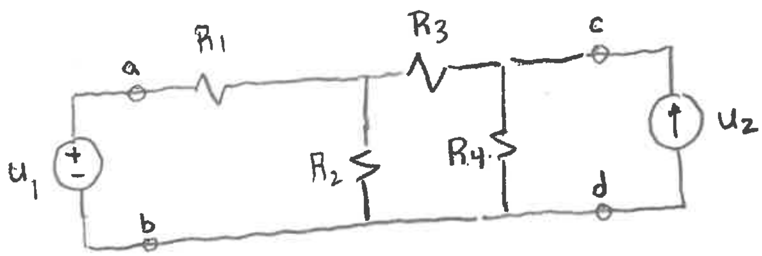
[a] Find a Thevenin equivalent at y looking leftward.

[b] " " Norton " " " " "

Hint: Relate y to i

$y = \boxed{?}\, i + \boxed{?}$

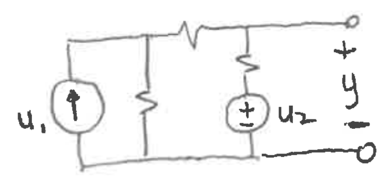
Consider the circuit:



[c] Find a Thevenin equivalent at u_1 (between a,b) looking rightward

[d] Find a thevenin equivalent at u_2 (between c,d) " leftward.

Consider the circuit (with all $R_i = 1$):



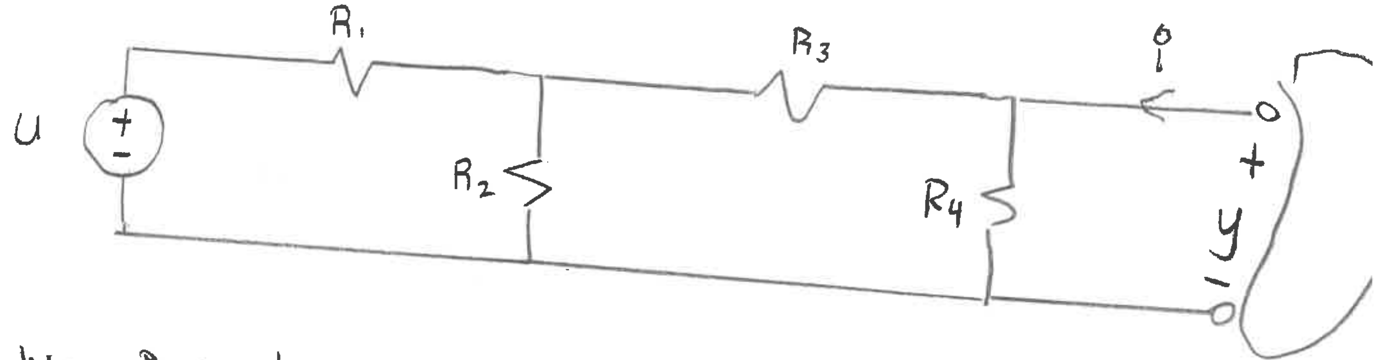
[e] Find a Thevenin equivalent at y looking leftward (using KVL, KCL, Ohm)

[f] Repeat [e] using source transformations Hint: Start at u_1 & proceed step-by-step rightward

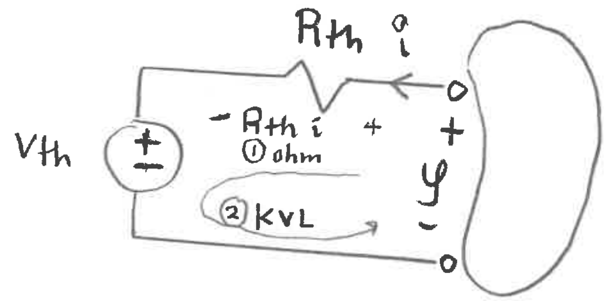
Addition to

$\boxed{\text{Example 18}}$ (Finding a Thevenin Equivalent via different Methods) 601

- Also useful for Thevenin Lab!!!



We wish to find a thevenin equivalent for the above circuit at y looking leftward:



② KVL: $y = R_{th}\, i + V_{th}$

How do we find R_{th} & V_{th}?

We shall do this using several methods.

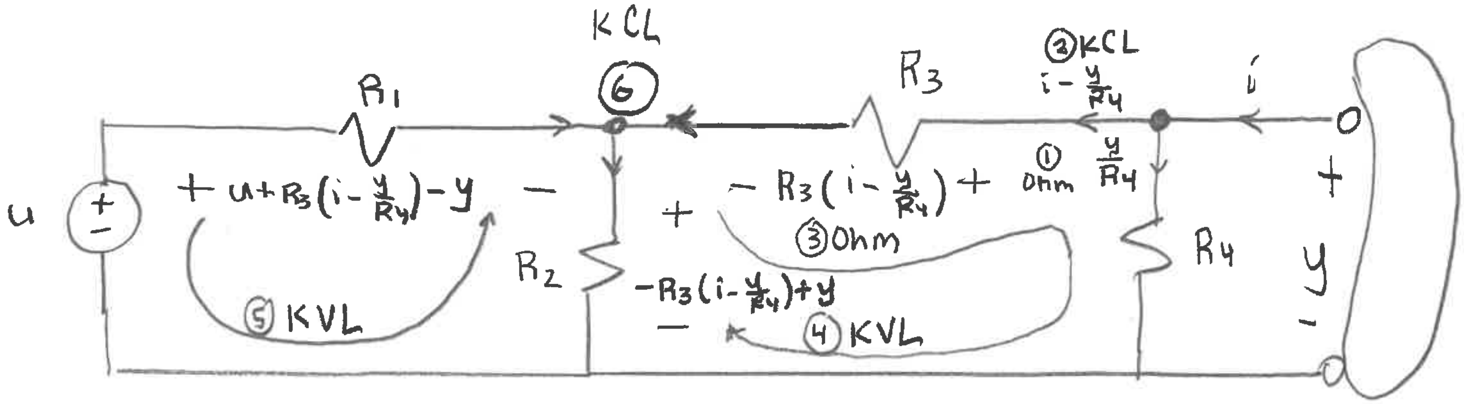
--- lets go ---

Addition to

Example 18 (Thevenin Equivalent: Different Methods)

Note: This is the most general method. It can always be applied... ...even when we have dependent sources!

a **Method 1** — use KVL, KCL, Ohm to relate y to i



⑥ KCL: (+ Ohm)

$$\left(\frac{u+R_3\left(i-\frac{y}{R_4}\right)-y}{R_1}\right)+\left(i-\frac{y}{R_4}\right)=\left(\frac{-R_3\left(i-\frac{y}{R_4}\right)+y}{R_2}\right)$$

"bucket" Algebra $\Longrightarrow$

$$u\left[\frac{1}{R_1}\right]+i\left[\frac{R_3}{R_1}+1+\frac{R_3}{R_2}\right]=y\left[\frac{R_3/R_4}{R_1}+\frac{1}{R_1}+\frac{1}{R_4}+\frac{R_3/R_4}{R_2}+\frac{1}{R_2}\right]$$

$$\Longrightarrow\ y=\underbrace{\left\{\frac{\left[1+R_3\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]}{(R_3/R_4+1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}}\right\}}_{R_{th}=\{\ \}}i+\underbrace{\left[\frac{\frac{1}{R_1}}{(R_3/R_4+1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}}\right]}_{V_{th}=[\ \ ]u}u$$

Lets do a little algebra to better understand the "form" of R_{th} & V_{th}.

Addition to

$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods) $\boxed{603}$

$\boxed{a}$ Using KVL, KCL, Ohm to find R_{th}, V_{th}:

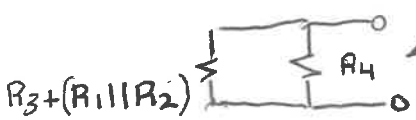
$$R_{th} = \frac{1 + R_3\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\left(\frac{R_3}{R_4}+1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right) + \frac{1}{R_4}} = \frac{1 + R_3\left(\frac{1}{R_1 \| R_2}\right)}{\left(\frac{R_3}{R_4}+1\right)\left(\frac{1}{R_1\|R_2}\right) + \frac{1}{R_4}}$$

$$\frac{R_4(R_1\|R_2)\times}{R_4(R_1\|R_2)\times} \quad \frac{R_4\left[R_1\|R_2 + R_3\right]}{(R_3+R_4) + R_1\|R_2}$$

Notes:
This is also useful for Thevenin Lab!

$$= \frac{R_4\left[R_3 + R_1\|R_2\right]}{R_4 + \left[R_3 + R_1\|R_2\right]}$$

$$\boxed{R_{th} = R_4 \,\|\, \left[R_3 + R_1\|R_2\right]}$$

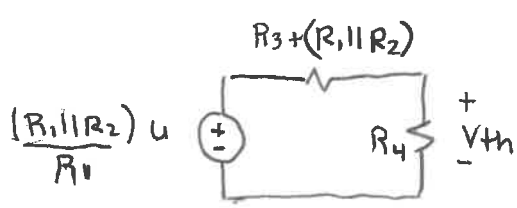


We'll see why R_{th} has this form soon!

$$V_{th} = \frac{\frac{u}{R_1}}{\left(\frac{R_3}{R_4}+1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}} = \frac{\frac{u}{R_1}}{\left(\frac{R_3}{R_4}+1\right)\left(\frac{1}{R_1\|R_2}\right)+\frac{1}{R_4}}$$

$$= \frac{\frac{R_1\|R_2}{R_1}u}{\left(\frac{R_3}{R_4}+1\right) + \frac{R_1\|R_2}{R_4}}$$

$$= \frac{R_4 \frac{R_1\|R_2}{R_1}u}{R_3 + R_4 + R_1\|R_2}$$



$$= \frac{\frac{(R_1\|R_2)u}{R_1}}{R_3 + R_1\|R_2}\left[\frac{R_4\left[R_3 + R_1\|R_2\right]}{R_4 + \left[R_3+R_1\|R_2\right]}\right]$$

$$\boxed{V_{th} = \left[\frac{\frac{(R_1\|R_2)u}{R_1}}{R_3 + R_1\|R_2}\right]\left\{R_4 \,\|\, \left[R_3 + R_1\|R_2\right]\right\}}$$

$$\frac{R_4}{R_4 + R_3 + R_1\|R_2}$$

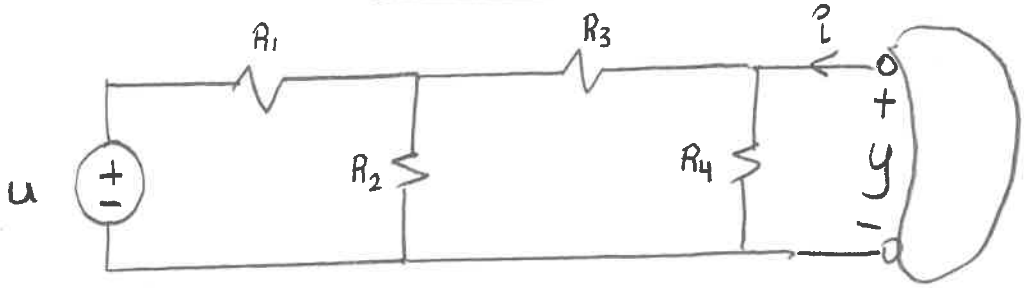
We'll also see why V_{th} has this form soon!

Addition to

$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods) $\boxed{604}$

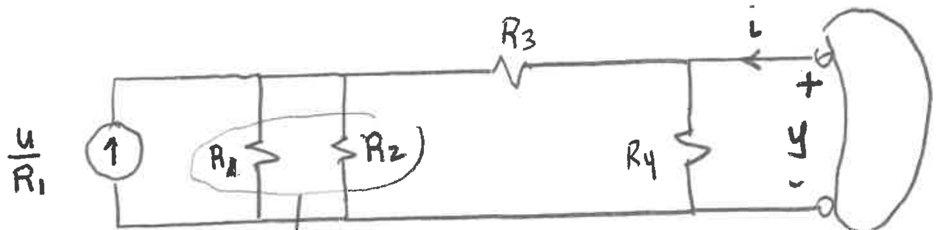
This method is "fairly straight forward" for the given circuit...

$\boxed{b}$ $\boxed{\text{Method 2}}$ $\boxed{\text{Find } V_{th} \text{ \& } R_{th} \text{ via repeated Source Transformations}}$ — The method also maximally illuminates the structure of V_{th} & R_{th}!



Note: This is also useful for Thevenin Lab!

Step 1: v to i transf

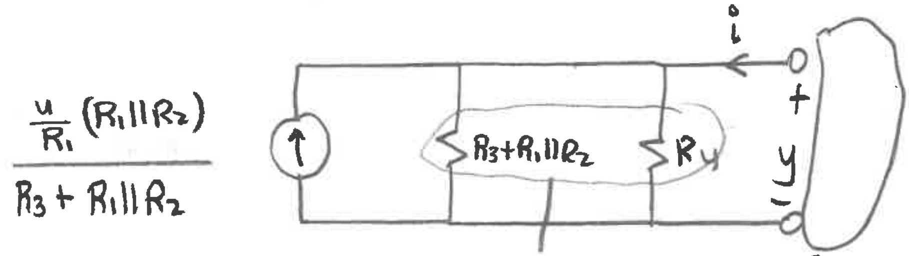


Step 2: $R_1 \| R_2$

Step 3: i to v transf

Step 4: $R_3 + R_1 \| R_2$ (series)

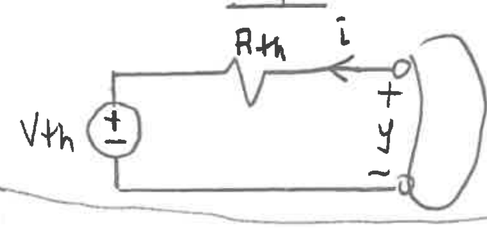
Step 5: v to i transf



Repeated Source Transf Approach has Shown why R_{th} & V_{th} have the form derived in $\boxed{a}$!

Step 6: $R_4 \| [R_3 + R_1 \| R_2]$

Step 7: i to v transf



$$\frac{R_4}{R_4 + R_3 + R_1 \| R_2}$$

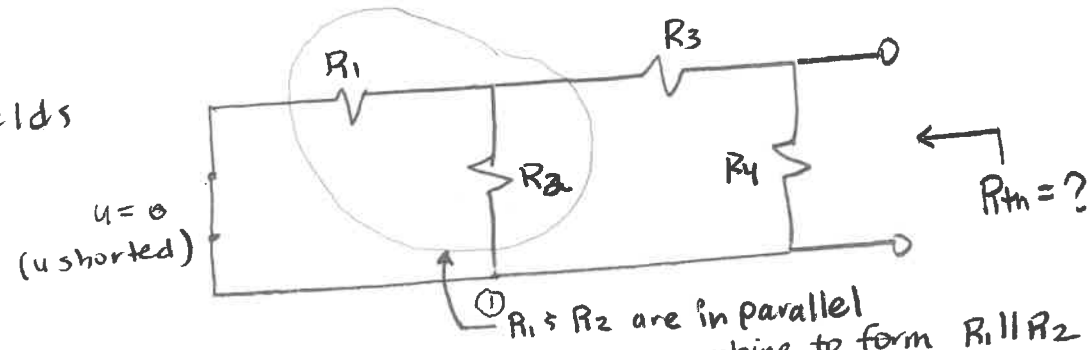
$$\boxed{\begin{aligned} R_{th} &= R_4 \| [R_3 + R_1 \| R_2] \\ V_{th} &= \left[\frac{\frac{u}{R_1}(R_1 \| R_2)}{R_3 + R_1 \| R_2}\right] [R_4 \| (R_3 + R_1 \| R_2)] \end{aligned}}$$

Addition to

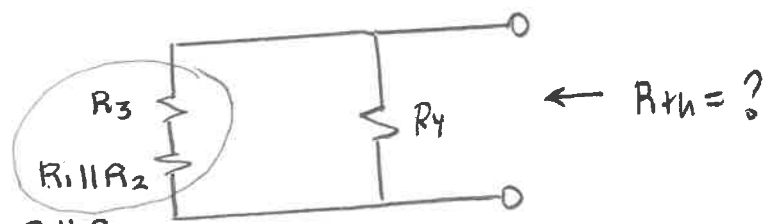
$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods)

$\boxed{c}$ Find R_{th} by setting $u=0$ & using series-parallel concepts.

letting $u=0$, yields



This can be rewritten as follows:



③ R_4 is in parallel with the combination $R_3 + R_1 \| R_2$

Note: This is also useful for Thevenin Lab!

$$\Longrightarrow \boxed{R_{th} = R_4 \,\|\, [R_3 + (R_1 \| R_2)]}$$

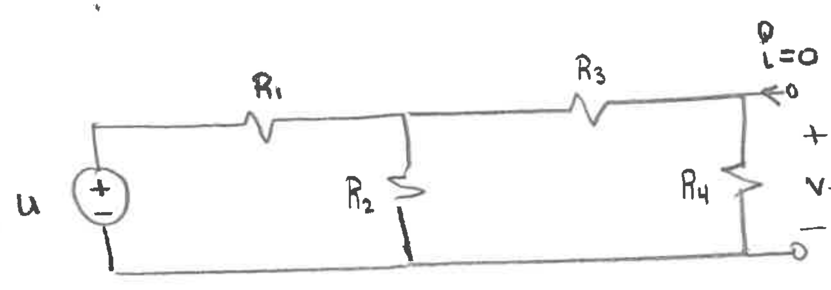
– as found in $\boxed{a}$ & $\boxed{b}$

Note: The series-parallel approach taken above in $\boxed{c}$ is (by far) the BEST APPROACH to find R_{th} (& understand its form) for the circuit being considered!

Addition to

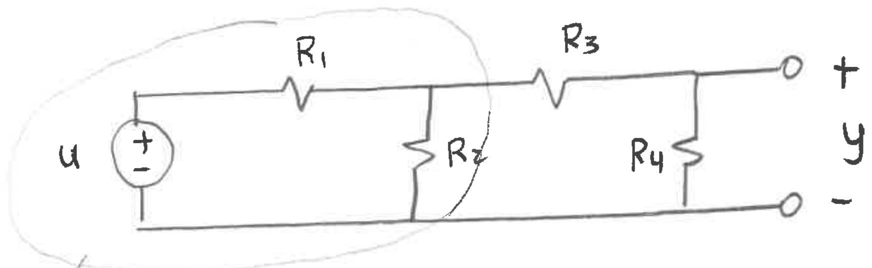
$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods) $\boxed{606}$

$\boxed{d}$ Find V_{th} by noting that it is the open circuit voltage $y_{oc} \triangleq y|_{i=0}$ (Also useful for Thevenin Lab!)

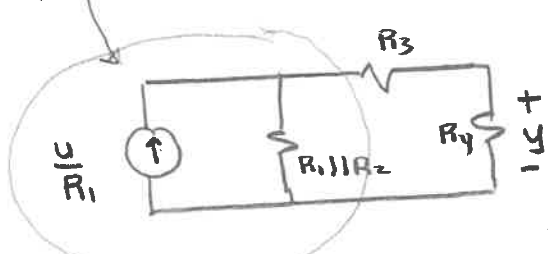


$V_{th} = y_{oc} \triangleq y|_{i=0}$

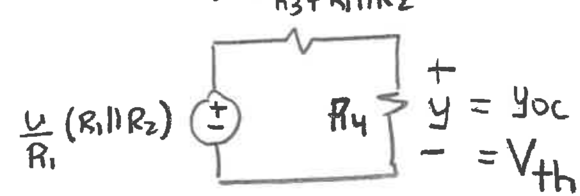
This is exactly like our standard (basic) problem of relating y to u:



While basic KVL, KCL, Ohm can be used, the source transformation method is maximally illuminating & fairly straight forward with some practice (of course!).

① 

$\xrightarrow{②}$



$\xrightarrow[\text{voltage division}]{③}$ $\boxed{V_{th} = \left[\frac{R_4}{R_3 + R_1\|R_2 + R_4}\right] \frac{u}{R_1}(R_1\|R_2)}$

↑ this is cleanest way to write V_{th}!!!

Finally, we note that this may be rewritten as

$$V_{th} = \frac{R_4\,[R_3+R_1\|R_2]}{R_3+R_1\|R_2+R_4}\left[\frac{\frac{u}{R_1}(R_1\|R_2)}{R_3+R_1\|R_2}\right] = \left[\frac{\frac{u}{R_1}(R_1\|R_2)}{R_3+R_1\|R_2}\right]\left[R_4\,\|\,[R_3+R_1\|R_2]\right]$$

as found in $\boxed{a}$, $\boxed{b}$

Addition to

$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods)

Also useful for Thevenin Lab!

$\boxed{e}$ Let $i_{sc} \triangleq i\big|_{y=0}$. Show that

$$R_{th} = -\left[\frac{y_{oc}}{i_{sc}}\right]$$

Also, show how to compute i_{sc}, compute it, & use y_{oc} & i_{sc} to compute $R_{th}(= -y_{oc}/i_{sc})$.

Since

$$y = R_{th}\, i + V_{th},$$

it follows that

$$V_{th} = y\big|_{i=0} \triangleq y_{oc}.$$

$$\Rightarrow \quad y = R_{th}\, i + y_{oc}$$

If $y = 0$, then $i = i_{sc} \triangleq i\big|_{y=0}$ &

$$0 = R_{th}\, i_{sc} + y_{oc}$$

or

$$R_{th}\, i_{sc} = -y_{oc}.$$

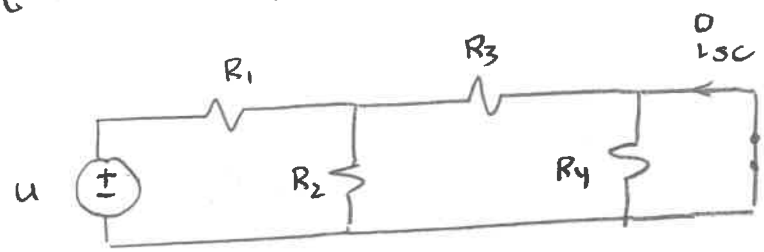
Hence,

$$R_{th} = -\left[\frac{y_{oc}}{i_{sc}}\right]$$

Addition to

Example 18 (Thevenin Equivalent: Different Methods)

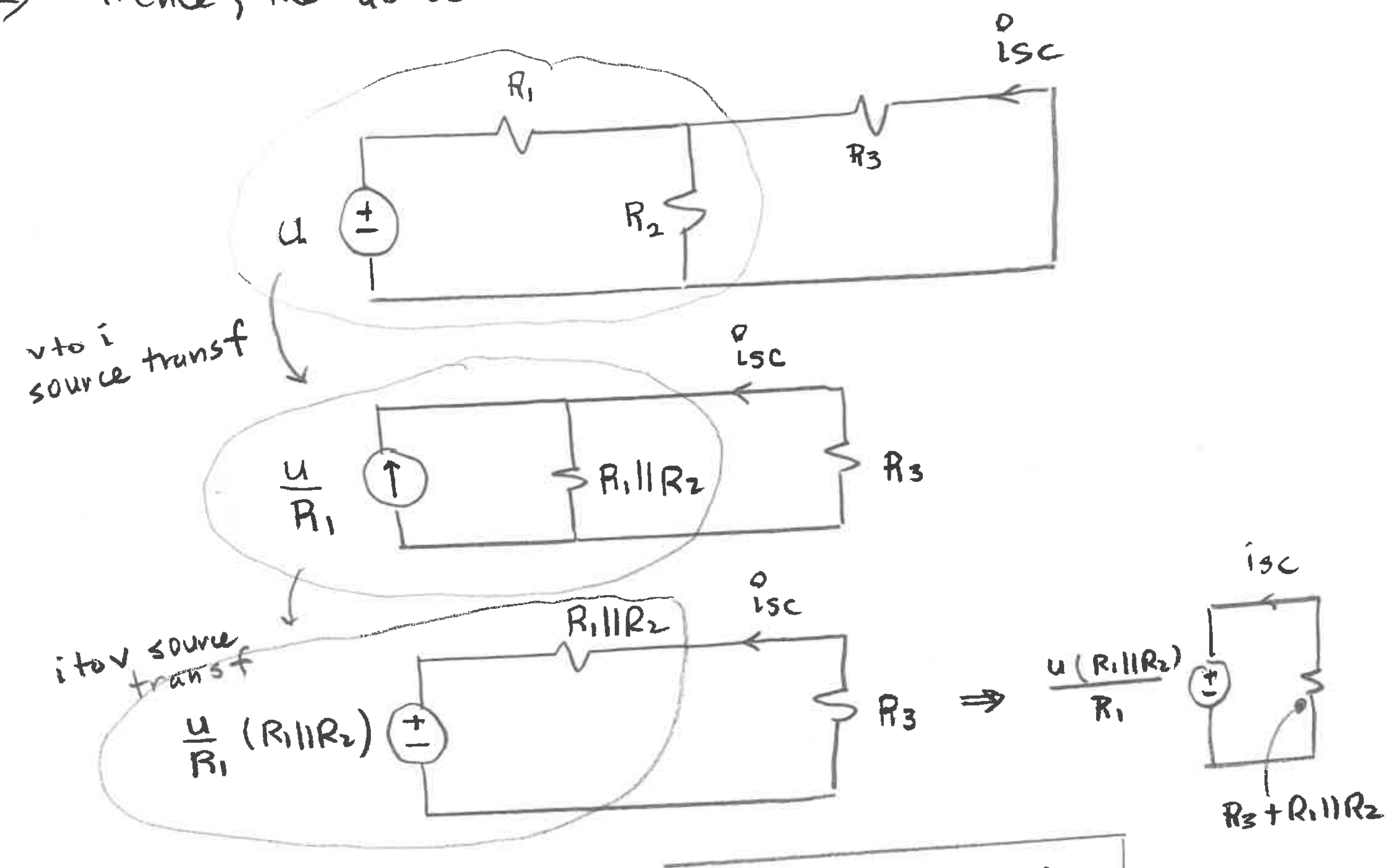
Find isc



y = 0 (y shorted)

From this, it follows that there is no voltage across R4 & hence no current through it!

⇒ Hence, the above reduces to



$$\Rightarrow \boxed{i_{sc} = -\frac{\frac{u(R_1 \| R_2)}{R_1}}{R_3 + R_1 \| R_2}}$$

Addition to

$\boxed{\text{Example 18}}$ (Thevenin Equivalent: Different Methods) $\boxed{609}$

Now recall from $\boxed{d}$ that

$$y_{oc} = V_{th} = \left[\frac{R_4}{R_4 + R_3 + R_1 \| R_2}\right] \frac{u(R_1 \| R_2)}{R_1}$$

Combining this with

$$i_{sc} = \frac{-\frac{u(R_1\|R_2)}{R_1}}{R_3 + R_1 \| R_2}$$

yields:

$$-\left[\frac{y_{oc}}{i_{sc}}\right] = \frac{\cancel{-}\left[\frac{R_4}{R_4 + R_3 + R_1\|R_2}\right]\cancel{\frac{u(R_1\|R_2)}{R_1}}}{\frac{\cancel{-}\cancel{u(R_1\|R_2)}}{\cancel{R_1}}\Big/ \left(R_3 + R_1\|R_2\right)}$$

$$= \frac{R_4\left[R_3 + R_1\|R_2\right]}{R_4 + R_3 + R_1\|R_2}$$

$$= R_4 \,\|\, \left[R_3 + R_1\|R_2\right]$$

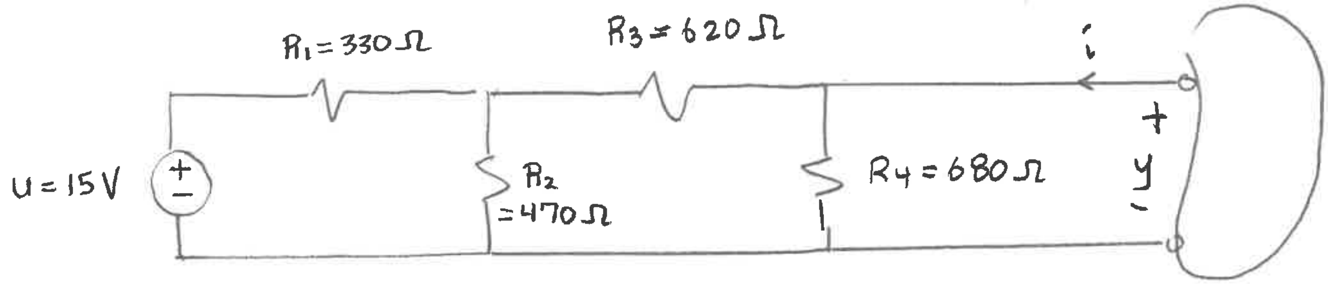
$= R_{th}$ as we found in $\boxed{a}, \boxed{b}, \boxed{c}$

Note:
In the lab, one can build the circuit, measure y_{oc} & i_{sc} & then use these to estimate R_{th}!

Note: The formula $R_{th} = -\left[\frac{y_{oc}}{i_{sc}}\right]$

Can only be used when we only have independent voltage & current sources in our circuit!!!
When the circuit has dependent sources, the formula cannot be used!!! --- One can still use KVL, KCL, Ohm as we did in $\boxed{a}$!!!

Thevenin Equivalent Lab Concepts



We wish to find a Thevenin equivalent at y looking leftward ; i.e. $\boxed{y = R_{th}\, i + V_{th}}$

Step 1 Compute ideal R_{th} & V_{th} using ideal numbers above & formulae we derived for R_{th} & V_{th}

i.e.

$$R_{th} = R_4 \parallel \left[R_3 + (R_1 \parallel R_2) \right]$$

$$V_{th} = \left\{ \frac{R_4}{[R_3 + R_1 \parallel R_2] + R_4} \right\} \frac{u}{R_1} (R_1 \parallel R_2)$$

Step 2: Measure u, R_1, R_2, R_3, R_4 with your multimeter.

Now calculate your expected R_{th} & V_{th} using the above formulae.

Note: Also calculate the short circuit current:

$$\boxed{i_{sc} = i \big|_{y=0} = \frac{-V_{th}}{R_{th}}}$$

using your calculated R_{th} & V_{th} !

Measurements to be Performed in Lab to Estimate R_{th} & V_{th}

I. Measure R_{th} directly with multimeter

II. Measure V_{th} " " "

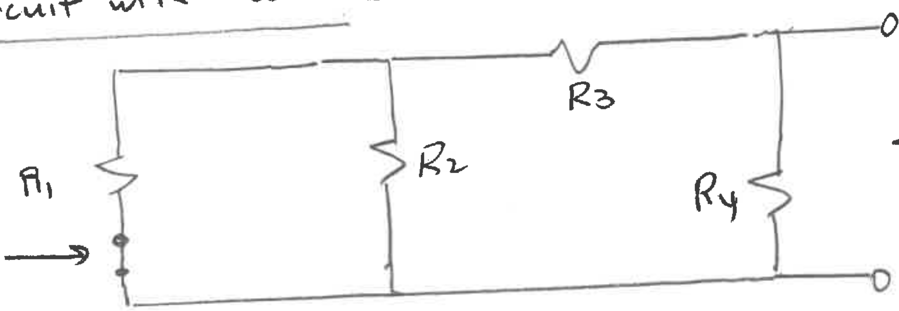
III. Measure $i_{sc} = y|_{i=0}$ (short circuit current) with multimeter.

How do we perform these measurements?

I. Measurement 1: Measure R_{th} directly with multimeter

Build circuit with $u = 0$:

set $u=0$ →



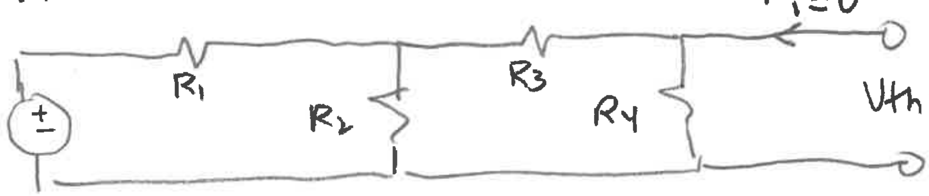
← Measure R_{th} directly with multimeter

- Compare this R_{th} to that found in step 2 above (page 1)

II. Measurement 2: Measure $V_{th} = y|_{i=0} = y_{oc}$ directly with multimeter

Build circuit & measure output voltage: $i=0$

$u = 15V$



open circuit voltage

$+$

$V_{th} = y|_{i=0} = y_{oc}$

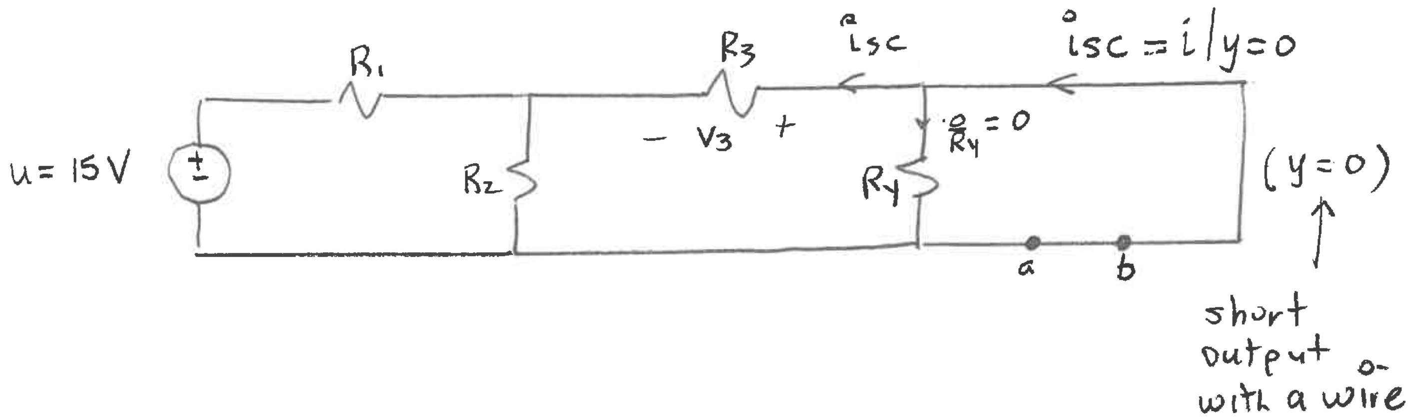
$-$

– compare this V_{th} to that found in step 2 above (page 1)

III Measurement 3: Measure $i_{sc} = i|_{y=0}$ directly with multimeter

short circuit current

Build circuit with $y = 0$ (ie short output):



Can measure i_{sc} with your multimeter (multimeter must be connected in series between points a & b!!!)

<u>Note:</u> Since voltage across R_y is zero, no current flows through R_y (by Ohm!)

If you measure V_3 with your multimeter, then you can estimate i_{sc} as:

$$i_{sc} = \frac{V_3}{R_3}$$

– compare this with what you got on page 1 step 2!!!

- Now estimate R_{th} using

$$R_{th} = \frac{-V_{th}}{i_{sc}}$$

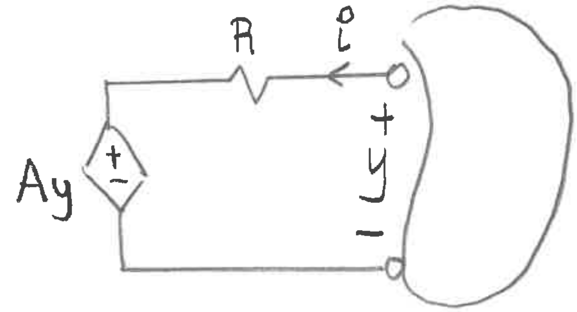
from $\boxed{II}$ on page 2

from $\boxed{III}$ on page 3

- compare this with what you got on page 1 step 2 !

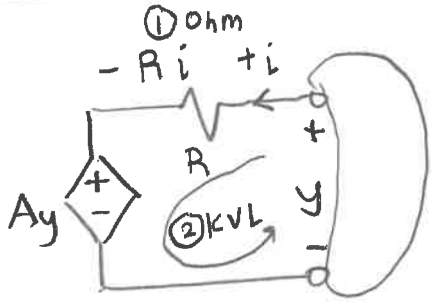
Example 19 (Dependent Sources Can Result In Negative Resistance!!!)

(MORE COOLNESS!!!)



a Relate y to i

① By Ohm



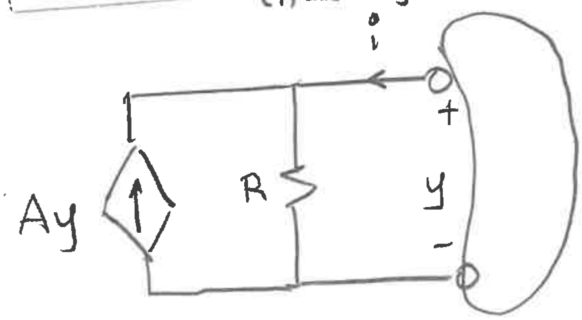
② By KVL: $y = Ri + Ay \overset{\text{algebra}}{\Longrightarrow} y(1-A) = Ri$

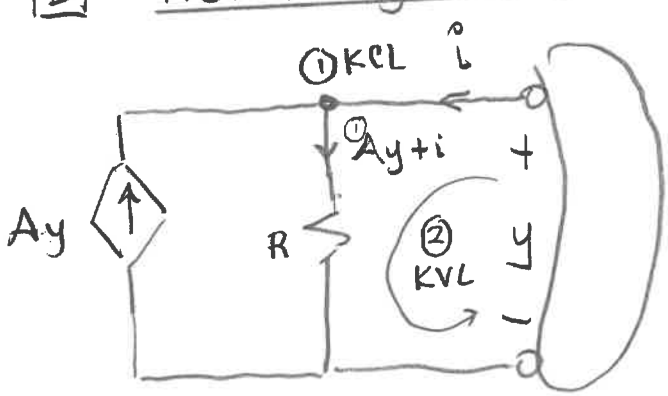
$$\Longrightarrow y = \left[\frac{R}{1-A}\right] i$$

$$\boxed{R_{th} = \frac{R}{1-A}}$$

Hence, $\boxed{R_{th} < 0 \overset{\text{iff}}{\underset{\text{(if and only if)}}{\Longleftrightarrow}} A > 1}$!

b Relate y to i





② KVL (+Ohm): $y = R(Ay + i) = Ri + RAy$

$$\Rightarrow y(1-RA) = Ri \Longrightarrow \boxed{y = \left[\frac{R}{1-RA}\right] u}$$

Example 19 (Dependent Sources Can Result In Negative Resistance !!!)

Since $y = \left[\frac{R}{1-RA}\right] i$

$$\boxed{R_{th} = \frac{R}{1-RA}}$$

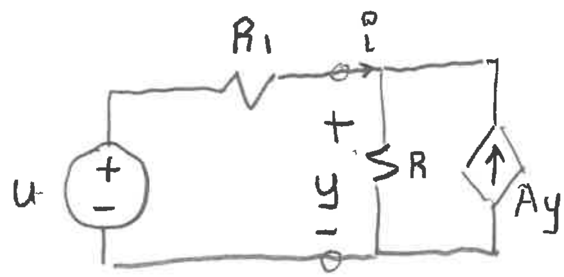
Hence,

$$\boxed{R_{th} < 0 \xleftrightarrow{\text{iff}} A > \frac{1}{R}}$$

(if and only if)

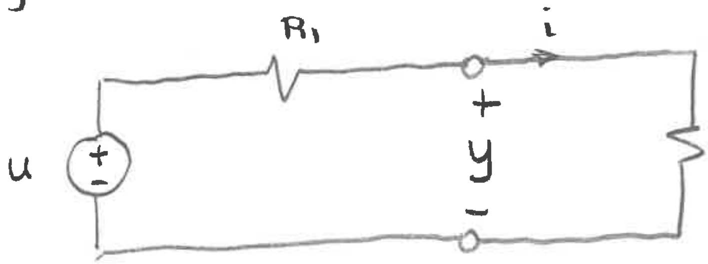
Now lets see how the above can impact a circuit.

Now consider the circuit



[c] Find a Theven equivalent from y looking rightward.

From [b]; it follows that



$R_{th} = \frac{R}{1-RA}$

[d] Relate y to u (Hint: This is a simple voltage divider ... see Example 2!)

From Example 2 on voltage division, we have

$$y = \left[\frac{R_{th}}{R_1 + R_{th}}\right] u$$

Example 19 (Dependent Sources Can Result In Negative Resistance !!!)

e In d, find R_{th} & A such that $y = -gu$

$$y = \left[\frac{R_{th}}{R_1 + R_{th}}\right] u = -gu \Rightarrow \frac{R_{th}}{R_1 + R_{th}} = -g$$

$$\Rightarrow R_{th} = -gR_1 - R_{th}\, g$$

$$\Rightarrow R_{th}(1+g) = -gR_1$$

$$\Rightarrow \boxed{R_{th} = \frac{-gR_1}{1+g}}$$

But $R_{th} = \dfrac{R}{1-RA}$

$$\Rightarrow R_{th} - R_{th} RA = R$$

$$\Rightarrow R_{th} RA = R_{th} - R$$

$$\Rightarrow A = \frac{R_{th} - R}{R_{th} R}$$

$$= \frac{1}{R} - \frac{1}{R_{th}}$$

$$= \frac{1}{R} - \frac{1}{\frac{-gR_1}{1+g}}$$

$$= \frac{1}{R} + \frac{1+g}{gR_1}$$

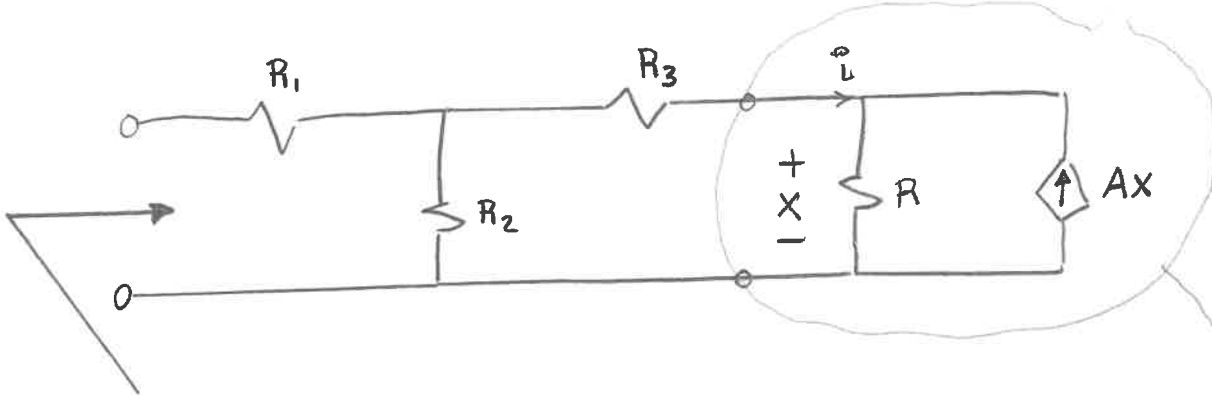
$$\boxed{A = \frac{1}{R} + \frac{1}{R_1}\left(1 + \frac{1}{g}\right)}$$

This is how one would (design) choose A so that $y = -gu$!

--- this is design!

Example 19

Consider the more complicated circuit:



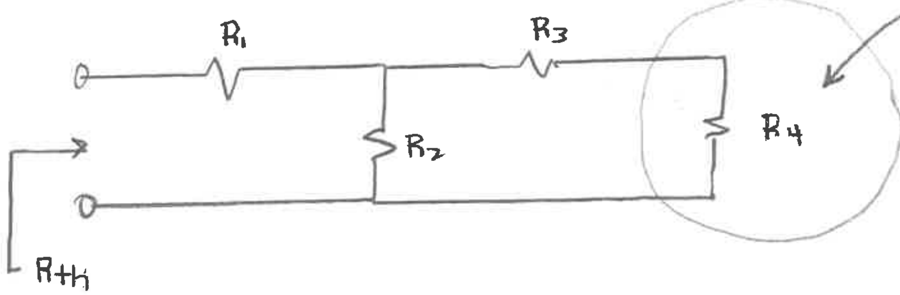
[f] Find Rth

1st we note (as in [b]) that

$$x = R(i + Ax)$$
$$= Ri + RAx$$
$$= \left(\frac{R}{1-RA}\right) i$$

$$R_4 \triangleq \frac{R}{1-RA}$$

Hence, the above circuit can be rewritten



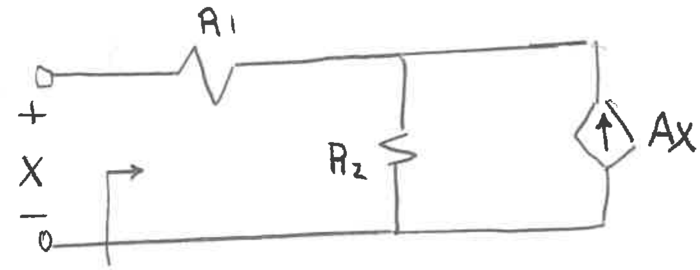
From this, it follows from series-parallel concepts that

$$\boxed{R_{th} = R_1 + R_2 \,\|\, (R_3 + R_4)}$$

where

$$\boxed{R_4 \triangleq \frac{R}{1-RA}}$$

9 Now consider a modified version of the circuit considered in **b**:



$R_{th} = ?$

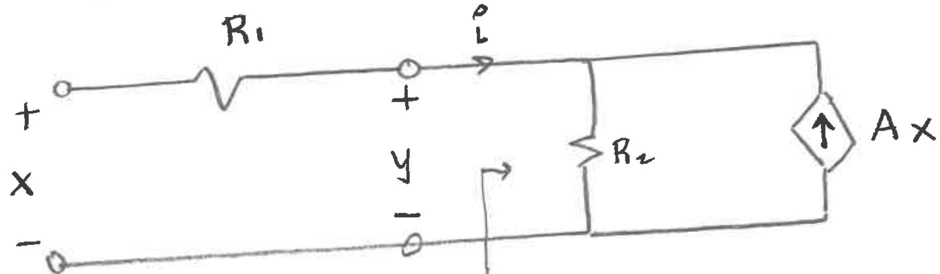
Note: If $R_1 = 0$, then we have the circuit in **b** &

$$R_{th} = \frac{R_2}{1 - R_2 A}$$

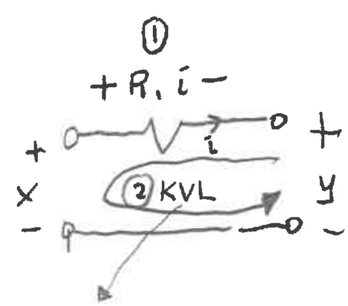
How do we find R_{th} when $R_1 \neq 0$?

We Know that $\boxed{R_{th} = R_1 + \hat{R}_2}$ for some $\hat{R}_2$

-- but how do we find $\hat{R}_2$?



$\hat{R}_2$; i.e. $y = \hat{R}_2 i$



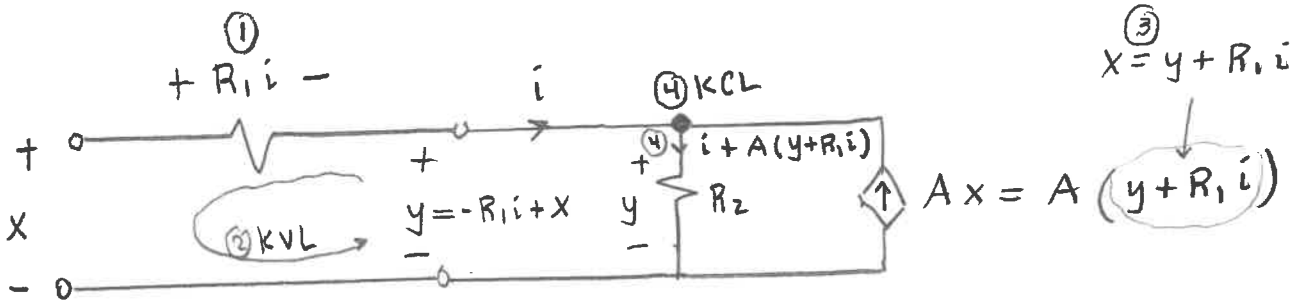
How can we relate y to i?

From ohm & KVL, we note that $y \overset{2}{=} -R_1 i + x$

Hence, $x = y + R_1 i$ --- Now we can ROCK! ---

Example 19

Using $x \overset{③}{=} y + R_1 i$, we can relate y to i (to get $\hat{R}_2$) by considering the following:



From this, it follows that

④KCL at top right node

⑤ Ohm across R_2

$$y \overset{⑤}{=} R_2 \left[i + A(y + R_1 i) \right]$$

$$= R_2 i + R_2 A y + R_2 A R_1 i$$

$$= R_2 [1 + A R_1] i + R_2 A y$$

$$= R_2 \left[\frac{1 + A R_1}{1 - A R_2} \right] i$$

$$= \hat{R}_2 i$$

$$\Longrightarrow \hat{R}_2 \triangleq R_2 \left[\frac{1 + A R_1}{1 - A R_2} \right]$$

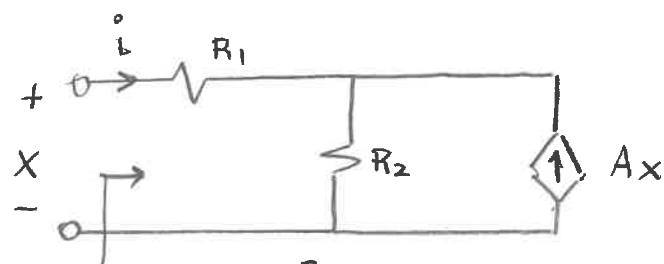
$$\Longrightarrow \boxed{\begin{aligned} R_{th} &= R_1 + \hat{R}_2 \\ \hat{R}_2 &\triangleq R_2 \left[\frac{1 + A R_1}{1 - A R_2} \right] \end{aligned}}$$

Note: when $R_1 = 0$, $R_{th} = \hat{R}_2 = R_2 \left[\frac{1+0}{1 - A R_2} \right]$

$= \frac{R_2}{1 - A R_2}$ as in $\boxed{b}$!

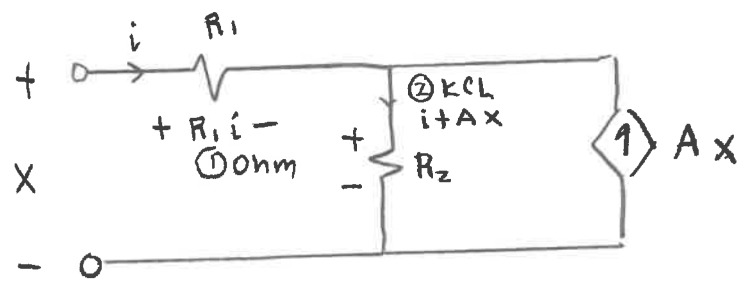
Example 19

Consider the same circuit as in $\boxed{9}$:



$R_{th} = ?$ (R_{th} was found in $\boxed{9}$ on pages 650 − 660)

$\boxed{h}$ Lets find R_{th} by relating x to i (i.e. $x = R_{th}\, i$)



③ KVL = (+Ohm)

$$x = R_1 i + R_2(i + Ax)$$

$$= (R_1 + R_2)i + A R_2 x$$

$$= \left[\frac{R_1 + R_2}{1 - AR_2}\right] i$$

using answer from $\boxed{9}$ → intelligent algebra

added to keep numerator the same

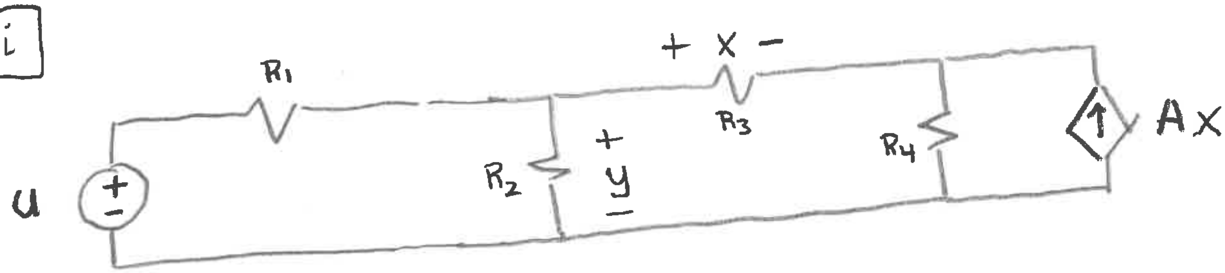
$$= \left[\frac{R_1(1 - AR_2) + R_2 + R_1 A R_2}{1 - AR_2}\right] i$$

$$R_{th} = R_1 + R_2\left[\frac{1 + AR_1}{1 - AR_2}\right]$$

... just as we got in $\boxed{9}$!

Example 19

i

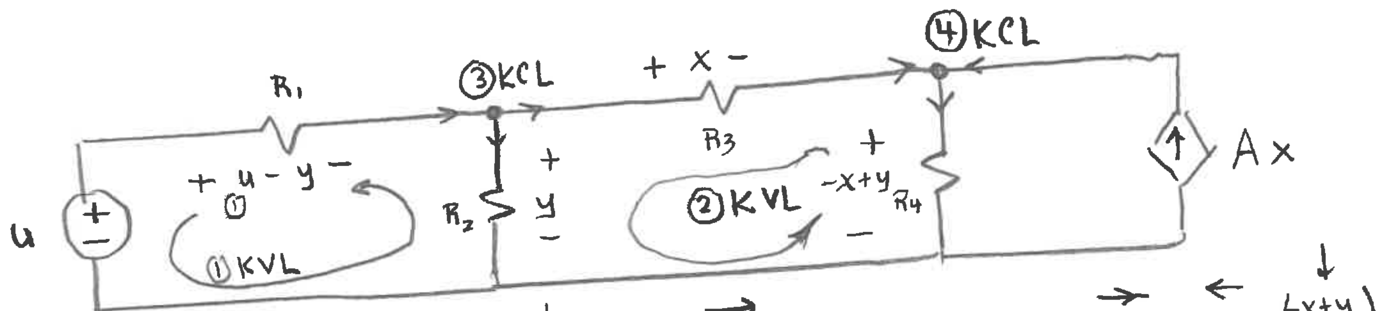


Relate y to u

Hint: Will need 2 eqs in the 2 unknowns (y, x).

Solution:

Lets take a straight KVL-KCL-Ohm approach:



③KCL (+ohm): $\left(\frac{u-y}{R_1}\right) = \left(\frac{y}{R_2}\right) + \left(\frac{x}{R_3}\right)$

algebra

$$\frac{u}{R_1} = y\left[\frac{1}{R_1} + \frac{1}{R_2}\right] + x\left[\frac{1}{R_3}\right]$$

④KCL (+ohm): $\left(\frac{x}{R_3}\right) + Ax = \left(\frac{-x+y}{R_4}\right)$

algebra

$$x\left[\frac{1}{R_3} + A + \frac{1}{R_4}\right] = \frac{y}{R_4}$$

$$\Longrightarrow x = \left[\frac{\frac{1}{R_4}}{A + \frac{1}{R_3} + \frac{1}{R_4}}\right] y$$

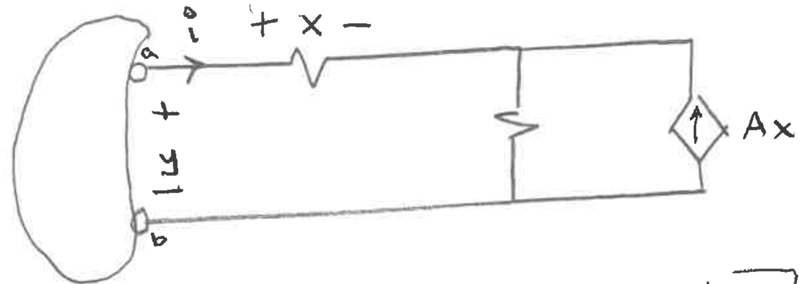
Substitute in for x

$$\frac{u}{R_1} = y\left[\frac{1}{R_1} + \frac{1}{R_2}\right] + \left[\frac{\frac{1}{R_4}}{A + \frac{1}{R_3} + \frac{1}{R_4}}\right]\frac{y}{R_3} = y\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{\frac{1}{R_3 R_4}}{A + \frac{1}{R_3} + \frac{1}{R_4}}\right]$$

$$\Longrightarrow y = \left[\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{\frac{1}{R_3 R_4}}{A + \frac{1}{R_3} + \frac{1}{R_4}}}\right] u \overset{R_i = 1}{=} \left[\frac{1}{2 + \left(\frac{1}{A+2}\right)}\right] u = \left(\frac{A+2}{2A+4+1}\right) u = \left[\frac{A+2}{2A+5}\right] u$$

Problem 19 (Assume all $R_i = 1$) 690

[a] Find a Thevenin equivalent at y looking rightward for

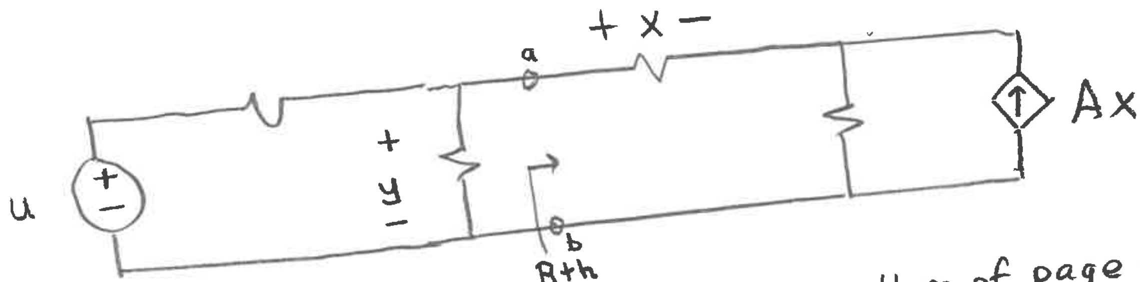


by relating y to i i.e. $y = \boxed{R_{th}}\, i$.

Hint: You'll need 2 eqs in the 2 unknowns y & x.
Get rid of x to get the desired relationship
relating y to i i.e. $y = \boxed{R_{th}}\, i$.

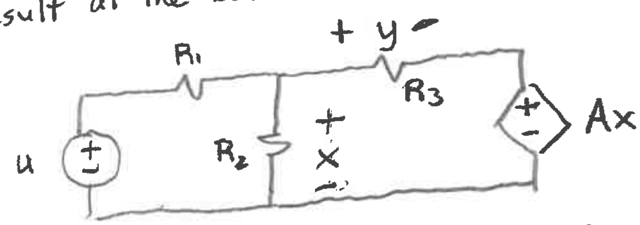
[b] Use the above R_{th}, parallel concepts, & voltage division to solve for y in the following circuit

(Assume all $R_i = 1$)



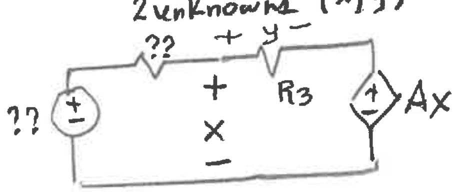
Note: You MUST get the result at the bottom of page 680!

Consider the following circuit



[c] Relate y to u using KVL, KCL, Ohm. Hint: Will need 2 eqs in 2 unknowns (x, y)

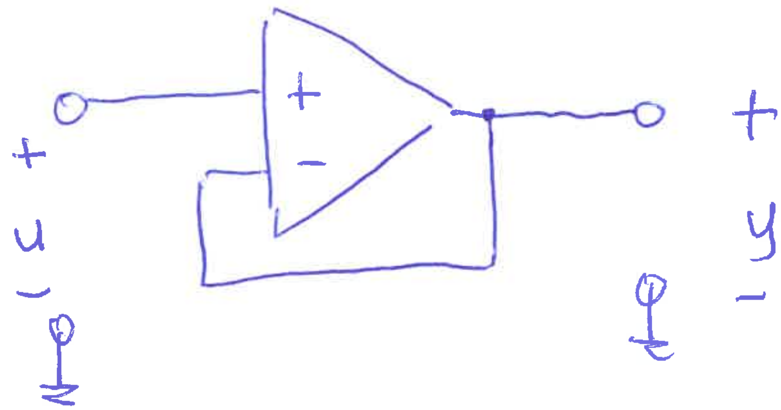
[d] Apply 2 source transformations to get →
Then relate y to u
(Show that [c] & [d] yield same result!)



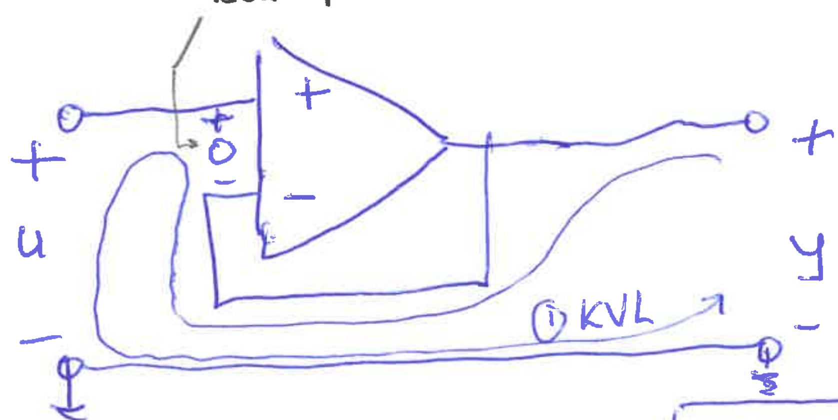
Example 20

Analysis of Buffer/Follower Amplifier with Standard op-amp Model

700



(a) Relate y to u using ideal op-amp assumptions

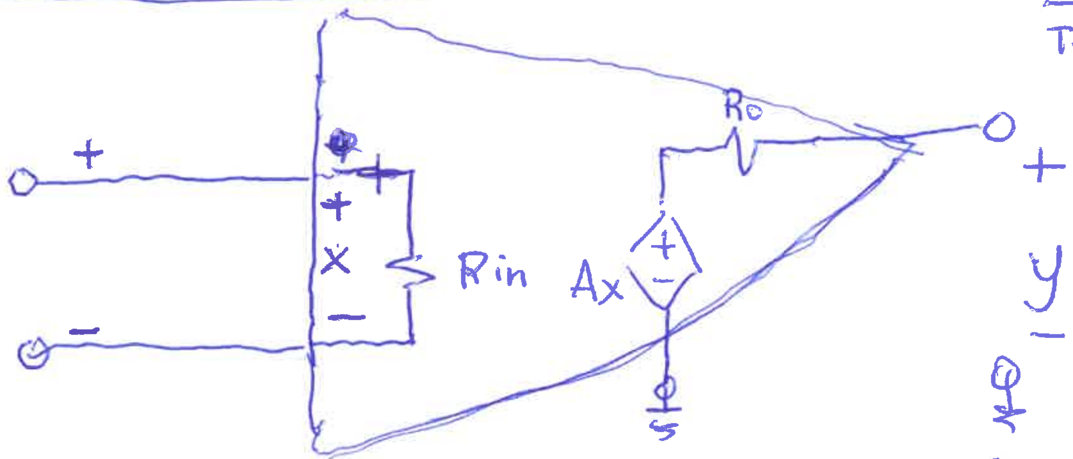


$$y = -0 + u \implies \boxed{y = u}$$

Ideal Buffer/Follower Input/Output Relation

Example 20 (Analysis of Buffer/Follower...)

Standard Model For an Op-Amp

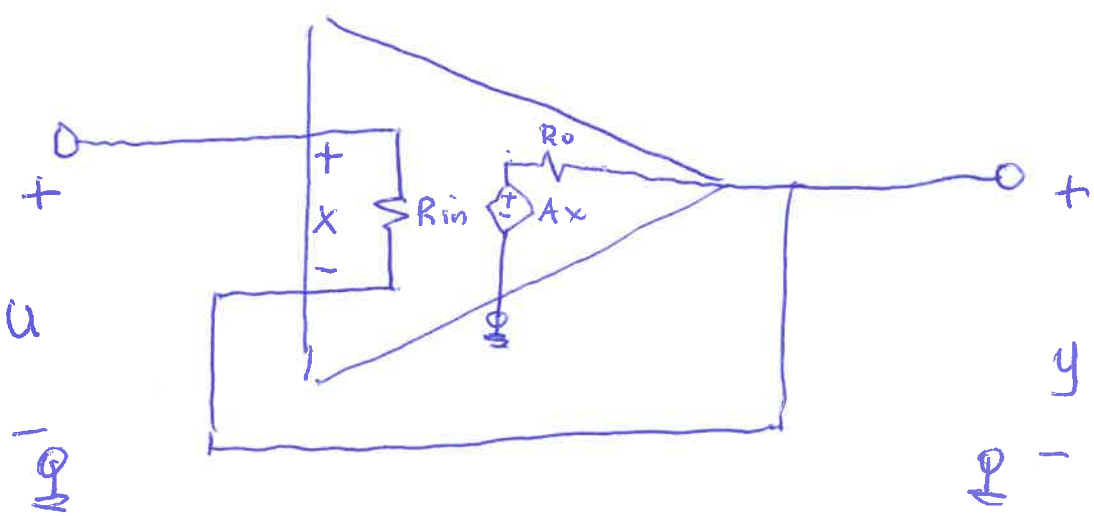


Note:
Typically,
Rin large
Ro small
A large

Rin - op-amp input resistance (impedance)
Ro - " " output " "

A- op-amp gain

Buffer/Follower Amplifier with Standard Model



Typical 741 Numbers:

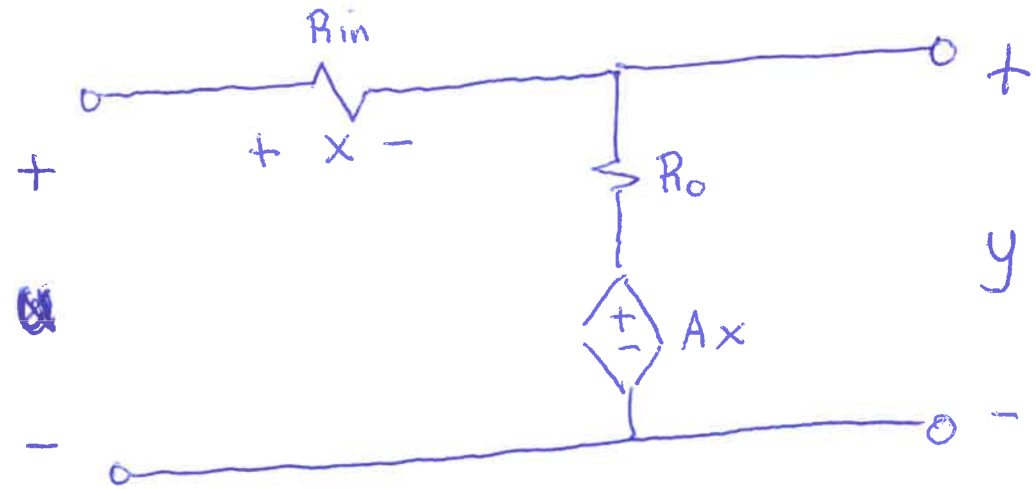
$R_{in} = 2M\Omega$
$R_o = 75\ \Omega$
$A = 200K$

Lets redraw the above.

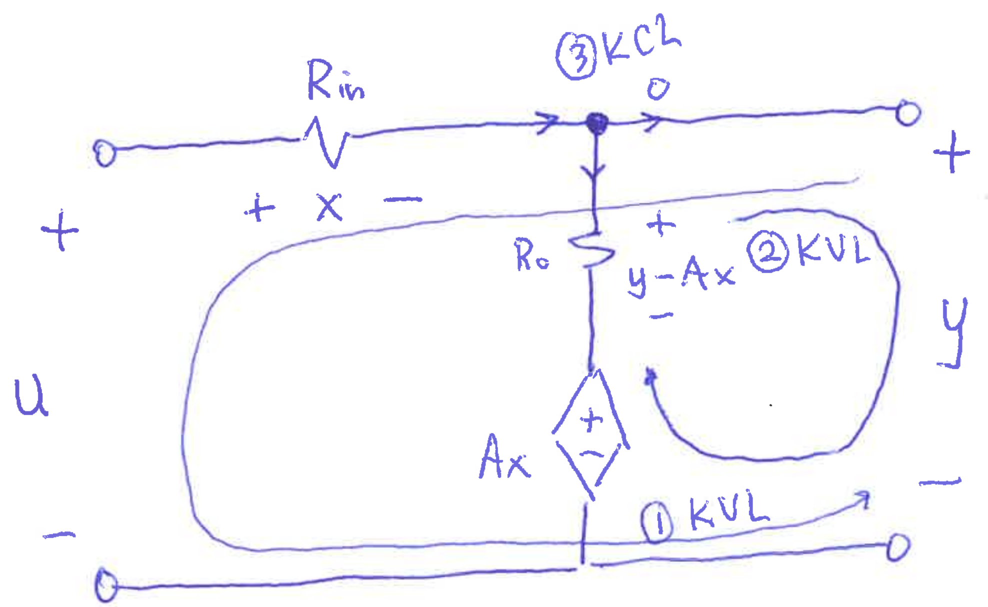
741 Reference: www.electronicshub.org/ic-741-op-amp-basics/

Example 20 (Analysis of Buffer/Follower...) 720

Buffer/Follower Amplifier with Standard Model



b Relate y to u using standard op-amp model



① KVL: $y = -x + u$

③ KCL (+Ohm) $\left(\frac{x}{R_{in}}\right) = \left(\frac{y-Ax}{R_o}\right) + \vec{0}$

Example 20

(Analysis of Buffer/Follower...)

We have

$$y = -x + u \qquad\qquad x\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] = \frac{y}{R_o}$$

$$\Downarrow$$

$$x = u - y$$

get rid of x to solve for y

$$[u-y]\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] = \frac{y}{R_o}$$

$$\Rightarrow \quad u\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] = y\left[\frac{1}{R_{in}} + \frac{A}{R_o} + \frac{1}{R_o}\right]$$

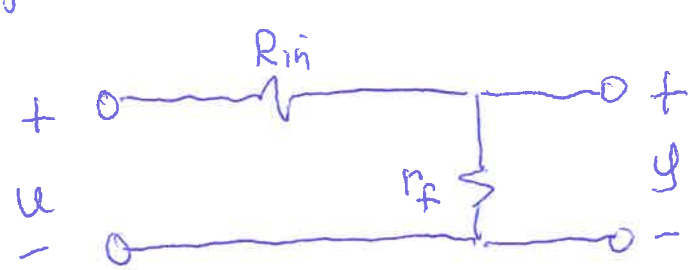
$$\xRightarrow{\times R_{in} R_o} \quad u\left[R_o + A R_{in}\right] = y\left[R_o + A R_{in} + R_{in}\right]$$

$$\Rightarrow \quad \boxed{\frac{y}{u} = \left[\frac{A R_{in} + R_o}{R_{in} + A R_{in} + R_o}\right]}$$

I/O relation for Buffer/Follower Amplifier with standard op-amp model

Cool Observation:

The above input-output relation has a simple voltage divider Interpretation:

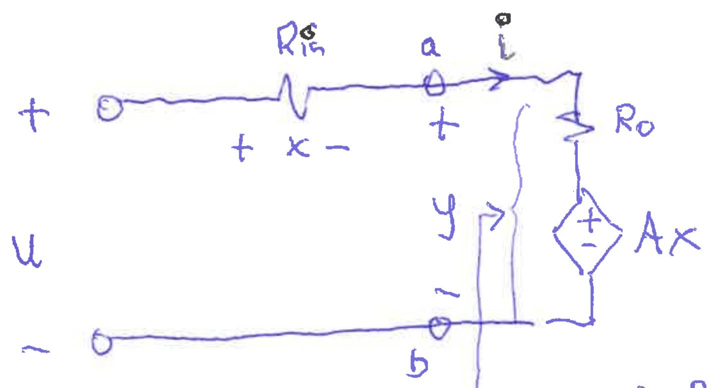


where $\boxed{r_f = A R_{in} + R_o}$

Example 20

(Analysis of Buffer/Follower)

Let try to understand why $r_f = A R_{in} + R_o$.



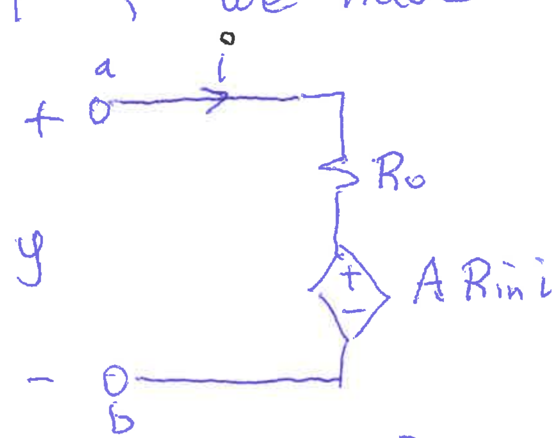
Why is it that this looks like

$r_f = A R_{in} + R_o$?

Lets find a thevenin equivalent between terminals a & b looking rightward.

We need to relate y to i!

By Ohm, $x = R_{in} i$ & we have



Hence, by KVL (& Ohm) we have

$$y = R_o i + A R_{in} i$$
$$= (A R_{in} + R_o) i$$
$$= r_f \, i \quad \text{where } r_f \triangleq A R_{in} + R_o$$

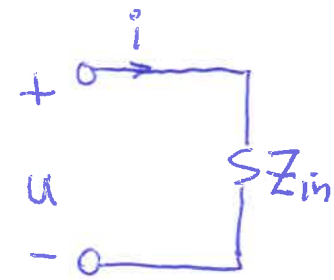
c Determine Input Impedance (resistance) Z_{in} at u looking rightward

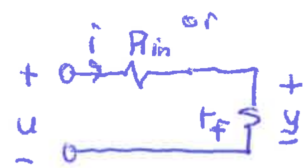
From the above analysis

$$Z_{in} = R_{in} + r_f$$

where

$$r_f = R_o + A R_{in}$$





d Let examine the I/O relationship derived in b as $R_{in} \to \infty$, $R_o \to 0$, & $A \to \infty$

"transfer function" from u to y

From b $H \triangleq \dfrac{y}{u} = \left[\dfrac{A R_{in} + R_o}{R_{in} + A R_{in} + R_o}\right]$, it follows that

$H(R_{in}, R_o, A)$

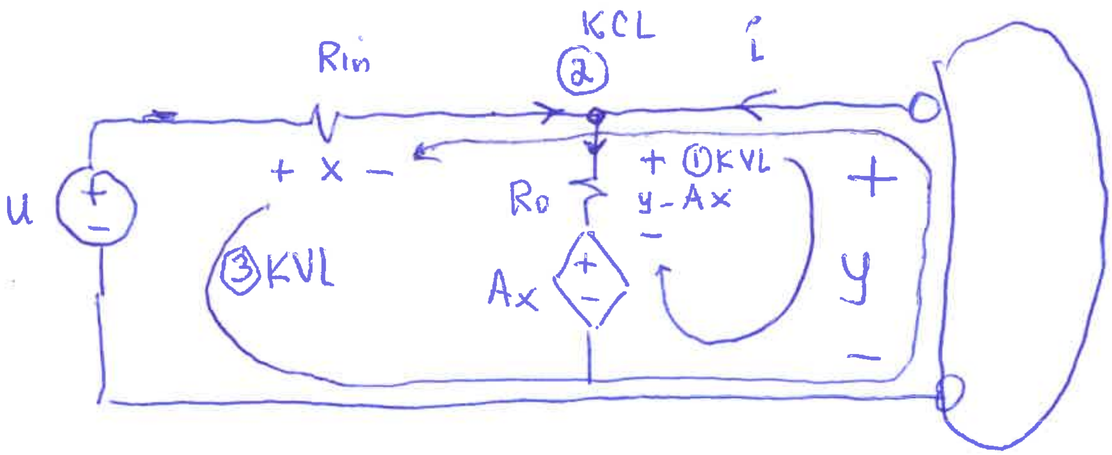
$$\lim_{R_o \to 0} H(R_{in}, R_o, A) = \frac{A R_{in}}{R_{in} + A R_{in}}$$

$$\lim_{\substack{R_{in} \to \infty \\ \text{or } A \to \infty}} H(R_{in}, R_o, A) = 1 \leftarrow$$

As R_{in} and/or $A \to \infty$, $H \to 1$ (the ideal buffer I/O relation!)

Example 20 (Analysis of Buffer/Follower ...) 760

e Find a thevenin equivalent at y looking leftward



i.e. we wish to relate y to i:

$$y = \boxed{R_{th\,out}}\ ?\ i + \boxed{V_{th\,out}}\ ?$$

② KCL (+ Ohm) =

$$\boxed{\left(\frac{x}{R_{in}}\right)^{\text{ohm}} + i = \left(\frac{y - Ax}{R_o}\right)^{\text{ohm}}}$$

or

$$x\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] + i = \frac{y}{R_o}$$

③ KVL: $x = u - y \longrightarrow$

$$[u-y]\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] + i = \frac{y}{R_o}$$

$\xRightarrow{\text{algebra}}$

$$u\left[\frac{1}{R_{in}} + \frac{A}{R_o}\right] + i = y\left[\frac{1}{R_{in}} + \frac{A}{R_o} + \frac{1}{R_o}\right]$$

Example 20 (Analysis of Buffer/Follower...)

$\times R_{in} R_o \Longrightarrow \quad u\left[R_o + A R_{in}\right] + R_{in} R_o \mathring{i} = y\left[R_o + A R_{in} + R_{in}\right]$

$$\Longrightarrow \quad y = \left[\frac{R_{in} R_o}{R_{in} + R_o + A R_{in}}\right]\mathring{i} + \left[\frac{R_o + A R_{in}}{R_{in} + R_o + A R_{in}}\right] u$$

$$\overset{\text{algebra}}{=} \left[\frac{R_{in} R_o}{R_{in}(1+A) + R_o}\right]\mathring{i} + \quad ''$$

$$= \left[\frac{R_{in}\left(\frac{R_o}{1+A}\right)}{R_{in} + \left(\frac{R_o}{1+A}\right)}\right]\mathring{i} + \quad ''$$

Thevenin equivalent at output looking leftward

$$\boxed{y = \left[R_{in} \,\Big\|\, \frac{R_o}{1+A}\right]\mathring{i} + \left[\frac{r_f}{R_{in} + r_f}\right] u}$$

"output impedance (resistance)"

$$R_{th_{out}} = Z_{out} = R_{in} \,\Big\|\, \frac{R_o}{1+A}$$

$V_{th_{out}}$ = I/O relation obtained in $\boxed{b}$ (see page 730)

Lets try to understand why $R_{th_{out}}$ at output has this $R_{in} \,\|\, \frac{R_o}{1+A}$ form!

Note:

$V_{th_{out}} = y_{oc}$ ← y open circuit

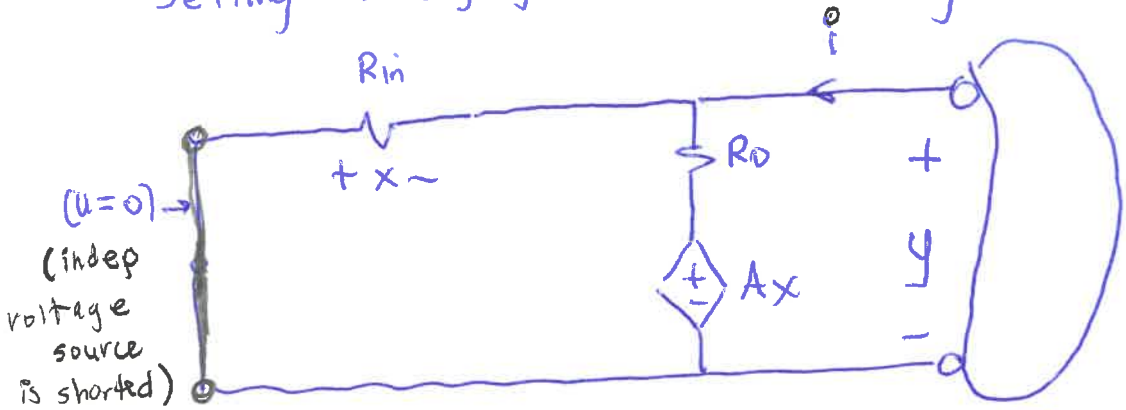
$\triangleq y\big|_{\mathring{i}=0}$

= the relation from $\boxed{b}$ on page 730

R_{th} can be found by setting $u=0$ & finding R_{th}...

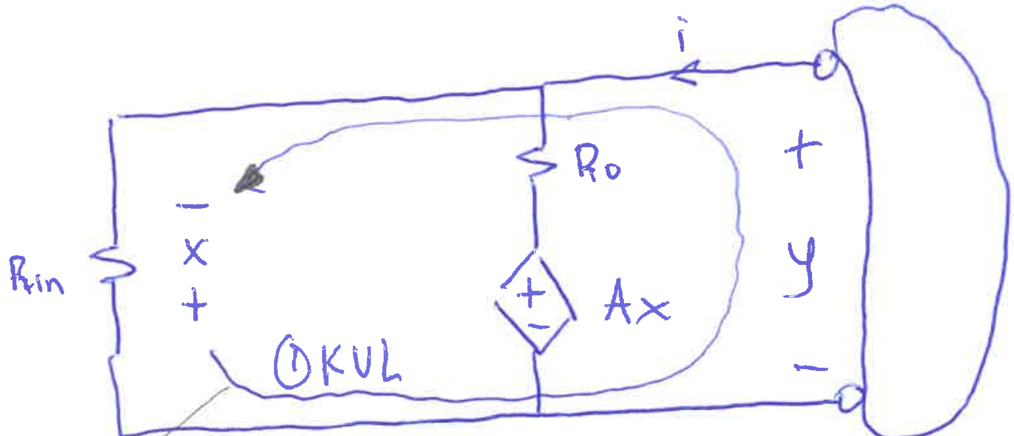
Example 20 (Analysis of Buffer/Follower...)

Setting input $u=0$, yields the following



Finding $R_{th_{out}}$ at y looking leftward (by 1st setting $u=0$)

or



From this we see that

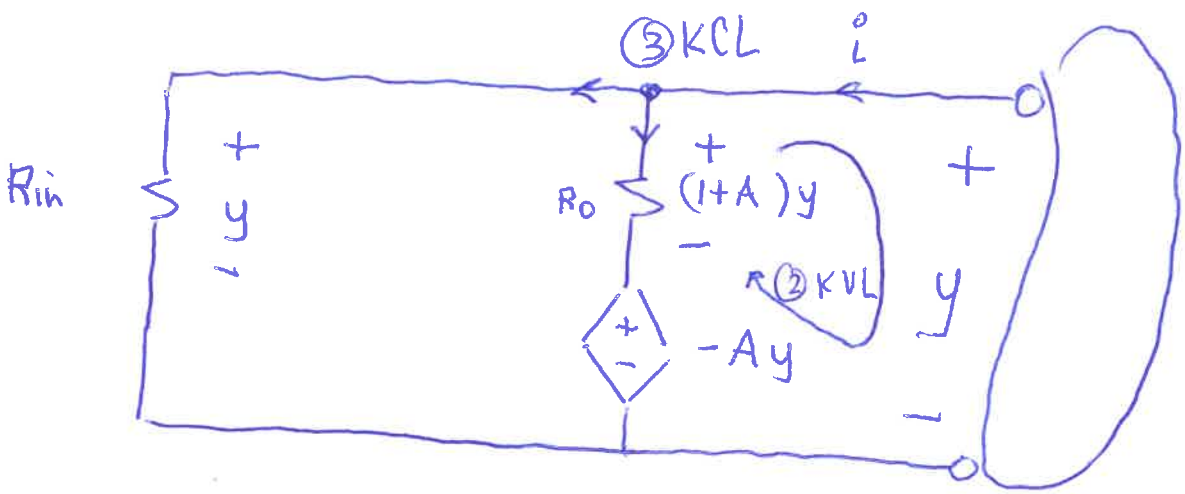
$R_{th_{out}} = Z_{out} =$ Rin in parallel with some resistance r_o that we need to find (we expect $r_o = \frac{R_o}{1+A}$ from page 770 !!!)

① KVL:

By KVL, we note that $\boxed{x = -y.}$

Applying this to the above circuit yields:

Example 20 (Analysis of Buffer/Follower....)



③KCL (1 ohm):

$$i = \frac{y}{R_{in}} + \frac{(1+A)y}{R_o}$$

$$= y\left[\frac{1}{R_{in}} + \frac{1}{\frac{R_o}{1+A}}\right]$$

$$= y\left[\frac{1}{R_{in} \parallel \frac{R_o}{1+A}}\right]$$

$$\Rightarrow y = \left(R_{in} \parallel \frac{R_o}{1+A}\right) i$$

$$r_o = \frac{R_o}{1+A}$$

Hence,

$$\boxed{R_{th\ output} = Z_{out} = (R_{in} \parallel r_o) \quad \text{where } r_o = \frac{R_o}{1+A}}$$

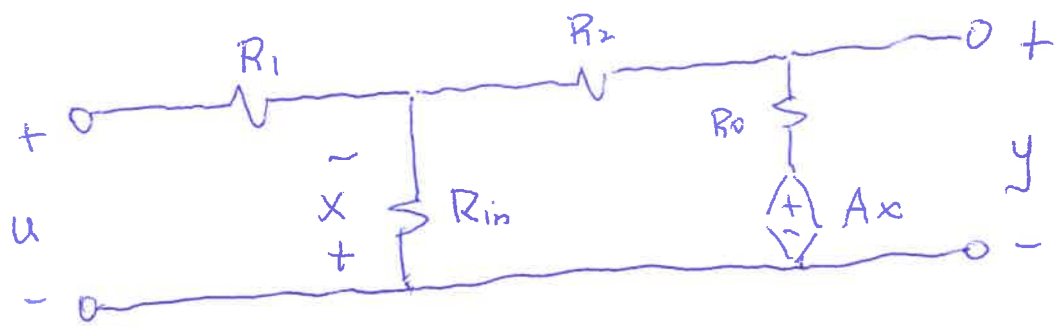
r_o has replaced

Problem 20

Analysis of Inverting & Non-Inverting Amplifiers with Standard Op-Amp Model

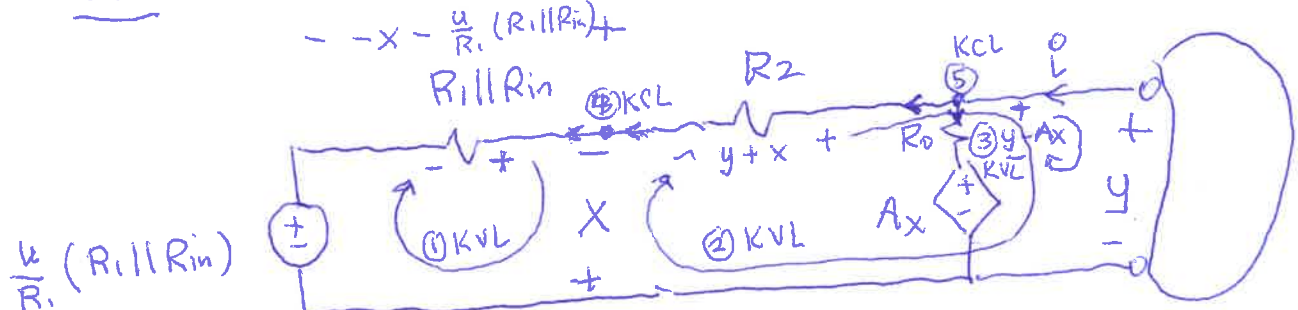
800

I Consider the following inverting amplifier circuit with a standard op-amp model:



a Find a Thevenin equivalent at output y looking leftward

Hint: Do 2 source transformations (v to i & then i to v) on u to get



Hints: Use KCL & Ohm at ④ to get 1st eq

" " " " " ⑤ " " 2nd eq

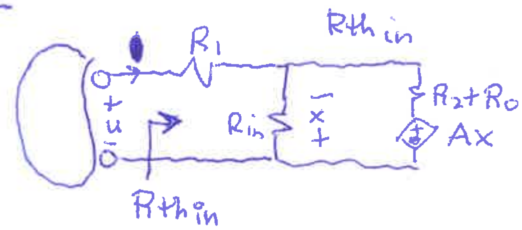
Get rid of x to relate y to i:

$y = \boxed{}\, i + \boxed{}$

$R_{th\,in}$ $V_{th\,in}$

b Find a thevenin equivalent at input u looking rightward

Hint: Consider



Problem 20 (... Inverting & Non-Inverting ...)

From the previous circuit, we see that

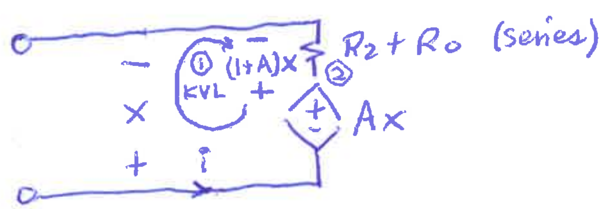
$$R_{th_{in}} = R_1 + R_{in} \,||\, r_o$$ where r_o needs to be found

Also $V_{th_{in}} = 0$ since there are no indep sources in our circuit!

How can you find r_o?

② Ohm: $(1+A)x = (R_2+R_o)\,i$

$$r_o = \frac{x}{i} = \,?$$



[c] How do your results in [a] & [b] change as

$R_o \to 0$

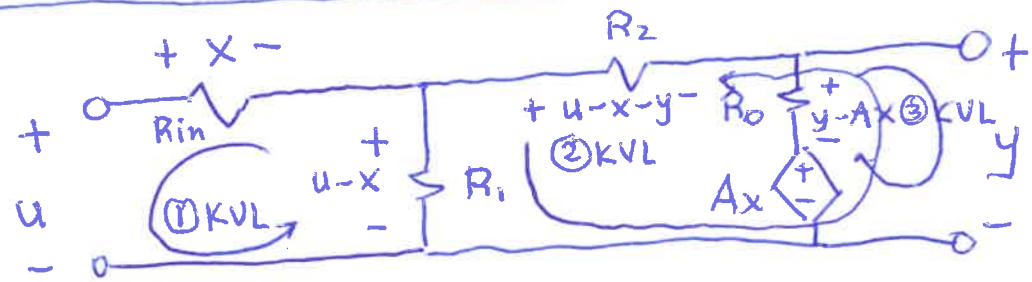
$R_{in} \to \infty$

$A \to \infty$

individually & then collectively

II

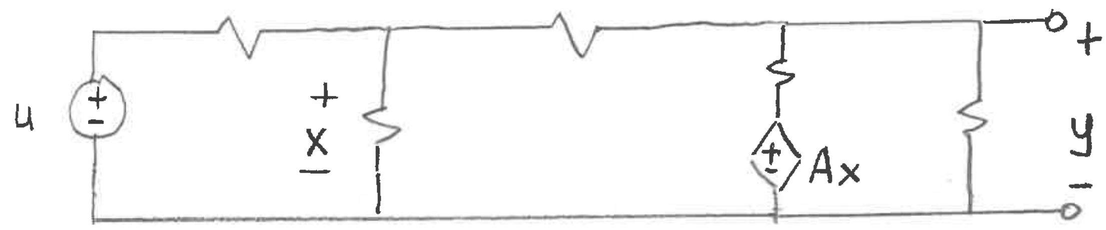
Consider the following non-inverting amplifier circuit with a standard op-amp model:



[d] [e] [f] Repeat [a], [b], [c] for this circuit.
↑[d] ↑[e] ↑[f]

Problem 20 (Circuits with Dependent Sources) 820

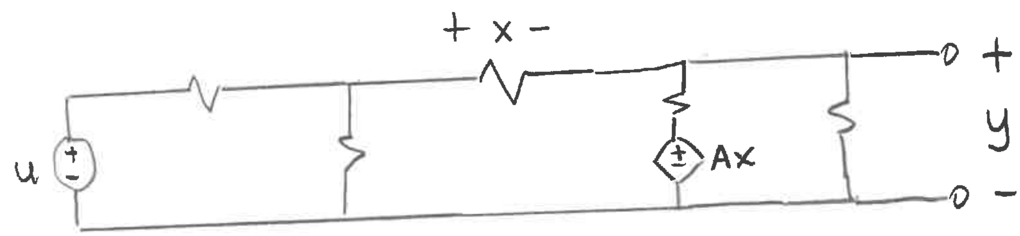
Consider the following circuit with all $R_i = 1$ ($\}$ $A > 0$) :



[a] Use KVL, KCL, Ohm to relate y to u; (i.e. $y = Hu$)
Find H ...

[b] For what values of A is $H < 0$?

Now suppose we have the following circuit (All $R_i = 1$, $A > 0$):



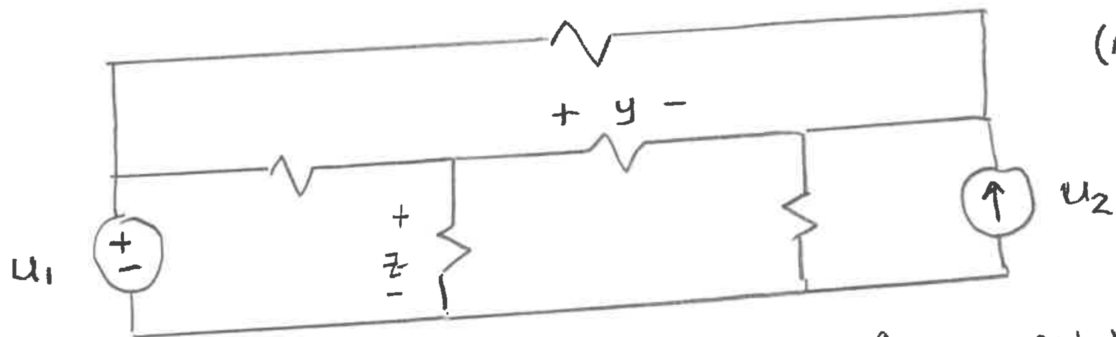
[c], [d] Repeat [a], [b] for this circuit.

Happy Circuiting!

Example 21 (Need Another Variable)

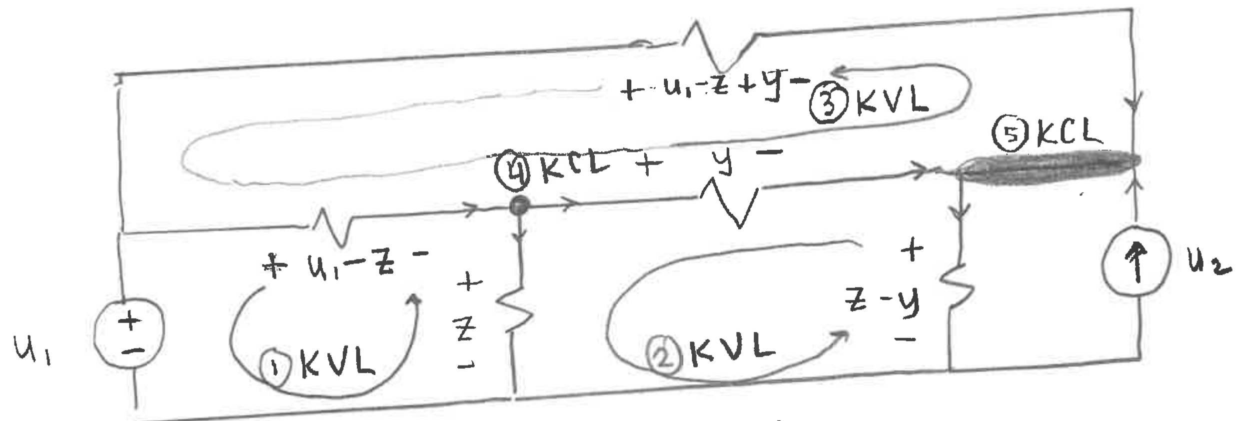
Consider the following circuit where the variable z has been introduced so that basic KVL, KCL, Ohm can be used to propagate y & z through the circuit:

(All $R_i = 1$)



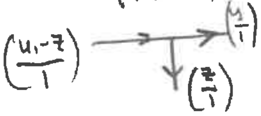
Note: Without introducing a new variable, in addition to y, we cannot propagate y around !!!

Relate y to u_1, u_2



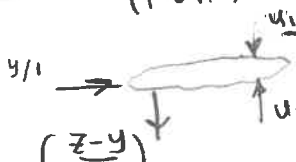
④ KCL (+Ohm) : $\left(\frac{u_1 - z}{1}\right) = \left(\frac{z}{1}\right) + \left(\frac{y}{1}\right) \xRightarrow{\text{algebra}} u_1 = 2z + y$

$2z = u_1 - y$



⑤ KCL (+Ohm) = $\left(\frac{y}{1}\right) + \left(\frac{u_1 - z + y}{1}\right) + u_2 = \left(\frac{z - y}{1}\right)$

$\Rightarrow 3y + u_1 + u_2 = 2z \longrightarrow 2z = 3y + \cancel{u_1} + u_2 = \cancel{u_1} - y$



$4y = -u_2$

Why is y indep of u_1? $\longrightarrow \boxed{y = -\left(\frac{1}{4}\right)u_2}$

$\boxed{\text{Example 21}}$ (Need Another Variable)

Lets compute all voltages in circuit:

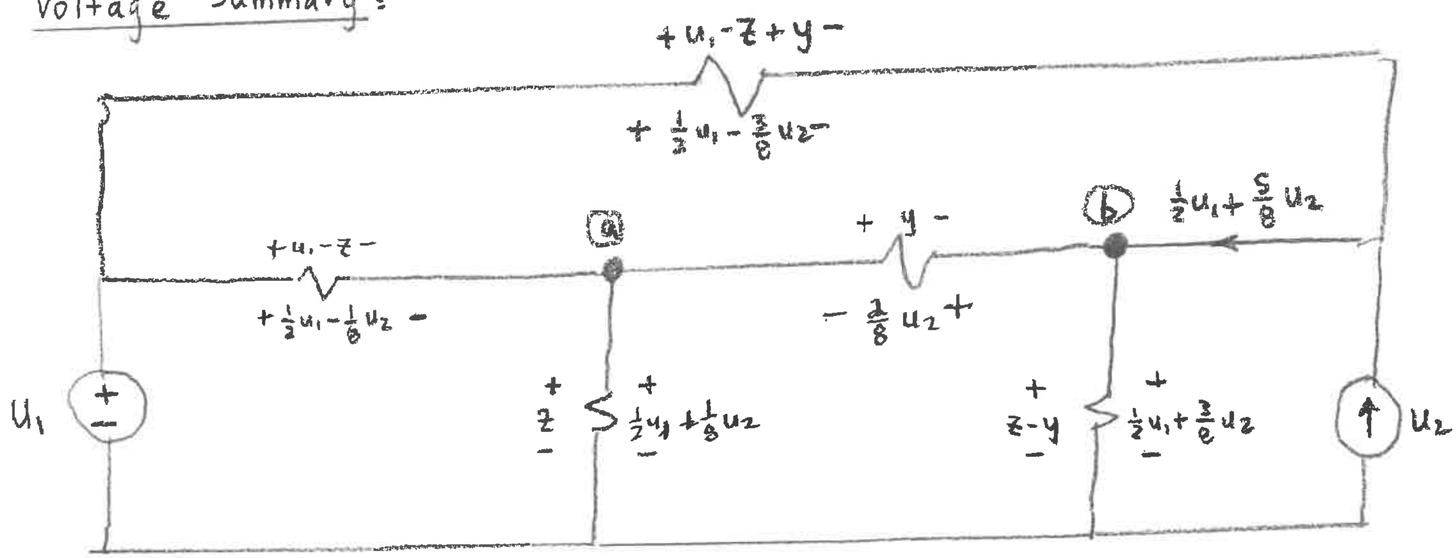
$$\Rightarrow \quad z = \tfrac{1}{2}u_1 - \tfrac{1}{2}y = \tfrac{1}{2}u_1 + \tfrac{1}{8}u_2 \quad \Rightarrow \quad \boxed{z = \tfrac{1}{2}u_1 + \tfrac{1}{8}u_2}$$

$$u_1 - z = u_1 - \tfrac{1}{2}u_1 - \tfrac{1}{8}u_2 \quad \Rightarrow \quad \boxed{u_1 - z = \tfrac{1}{2}u_1 - \tfrac{1}{8}u_2}$$

$$z - y = \tfrac{1}{2}u_1 + \tfrac{1}{8}u_2 + \underbrace{\left(\tfrac{1}{4}\right)}_{\frac{2}{8}}u_2 \quad \Rightarrow \quad \boxed{z - y = \tfrac{1}{2}u_1 + \tfrac{3}{8}u_2}$$

$$u_1 - z + y = \tfrac{1}{2}u_1 - \tfrac{1}{8}u_2 - \underbrace{\left(\tfrac{1}{4}\right)}_{\frac{2}{8}}u_2 \quad \Rightarrow \quad \boxed{u_1 - z + y = \tfrac{1}{2}u_1 - \tfrac{3}{8}u_2}$$

Voltage Summary:



Note: The above shows that KVL is satisfied around each loop & KCL is satisfied at each node.

A simple KCL at nodes ⓐ ⓑ shows why y is indep of u_1!

... the $\frac{1}{2}u_1$ flowing rightward into ⓐ gets completely directed downward
... none of it reaches y!

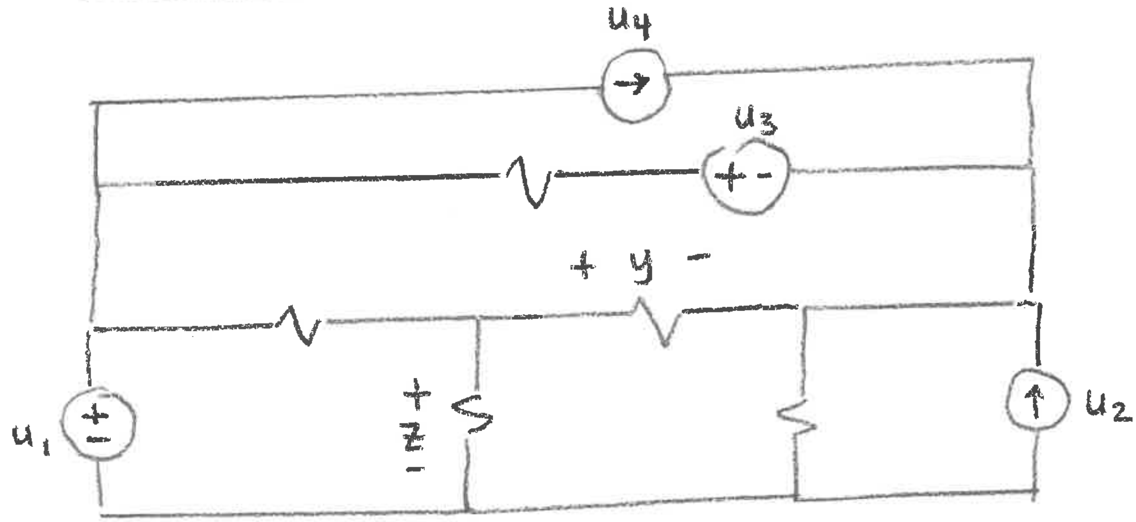
... the $\frac{1}{2}u_1$ flowing leftward into ⓑ gets completely directed downward
... none of it reaches y!

Note: Summaries like above can show us that KVL & KCL are in fact satisfied everywhere. They can also shed light on what is actually going on in the circuit.

Problem 21

(Need Another Variable)

Relate y to $u_{1,2,3,4}$ (assume all $R_i = 1$)



Introduction to Impedance Concepts

Impedance generalizes the concept of resistance.

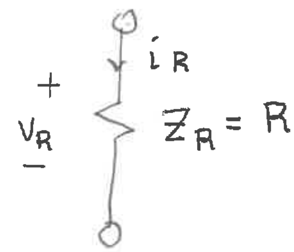
B60

Impedance of a Resistor

$$\boxed{v_R = R\, i_R}$$

(time-domain representation)

$$\updownarrow$$



$$V_R = R\, I_R$$

$$\boxed{Z_R = \frac{V_R}{I_R} = R}$$

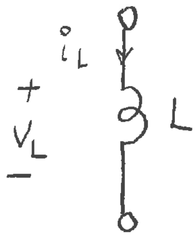
(impedance or Laplace s-domain representation)

Note:
Capital letters are often used for voltages & currents when working in the "impedance domain." (Laplace s-domain)

- We now introduce the impedance of an inductor & a capacitor.

- With such concepts, we will be able to treat inductors & capacitors (algebraically) just like they were resistors!

Impedance of an Inductor



$$\boxed{V_L = L\frac{di_L}{dt}}$$

(time-domain representation)

$$\boxed{\frac{d}{dt} \longleftrightarrow s}$$

$$V_L = L\,(s I_L)$$

we are simply replacing $\frac{d}{dt}$ with s!

$$= (sL)\, I_L$$

Note:
Capital letters are often used for voltages & currents when working in the "impedance domain"; (i.e. Laplace s-domain)

$$\boxed{Z_L = \frac{V_L}{I_L} = sL}$$

This is an s-domain equivalent (notational) representation for $V_L = L\frac{di_L}{dt}$!

(impedance or Laplace s-domain representation)

Note:

1) Z_L is "small" (near short; little "resistance" to current flow) when s is "small".

2) Z_L is "large" (nearly open; large "resistance" to current flow) when s is "large".

3) $\boxed{\text{Current } i_L \text{ through an inductor cannot (real) jump!}}$

Why? Such a jump would result in an infinite (∞) voltage V_L - This is NOT possible in the real world! A "large" V_L is possible, but NOT ∞ V_L!

$\boxed{\text{s should be thought of as representing a frequency.}}$

Think of (angular) frequency ω_0 in a sinusoid:

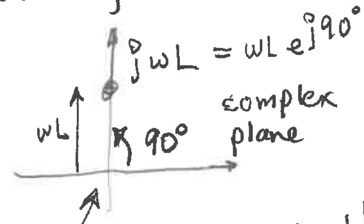
$A\sin(\omega_0 t + \theta)$

More on this later!!!

Note:

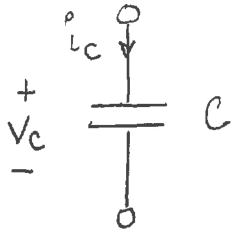
When $s = j\omega$, $Z_L = \underset{\text{rectangular form}}{j\omega L} = \underset{\text{polar form}}{\omega L e^{j90}} = |Z_L|\, e^{j\angle Z_L}$

$|Z_L| = \omega L$ $\angle Z_L = 90$



pictures are important!

Impedance of a Capacitor



$$\boxed{i_C = C\frac{dv_C}{dt}}$$

(time-domain representation)

$$\boxed{\frac{d}{dt} \longleftrightarrow s}$$

We are simply replacing $\frac{d}{dt}$ with s!

$$I_C = C\,(s V_C)$$

$$= (sC)\, V_C$$

$$\boxed{Z_C = \frac{V_C}{I_C} = \frac{1}{sC}}$$

(impedance or Laplace s-domain representation)

Note: Capital letters are often used for voltages & currents when working in the "impedance domain" (i.e. Laplace s-domain).

Note: 1) Z_C is large (nearly open; large "resistance" to current flow) when s is large.

2) Z_C is small (near short; small "resistance" to current flow) when s is small.

3) $\boxed{\text{voltage } V_C \text{ across a (real) capacitor cannot jump!}}$

Why? Such a jump would result in an infinite (∞) current i_C.

This is NOT possible in the real world!

A "large" i_C is possible, but NOT ∞ i_C!

$\boxed{\text{s should be thought of as representing a "frequency."}}$

Think of (angular) frequency ω_0 in a sinusoid:

$A \sin(\omega_0 t + \theta)$
cos

More on this later!!!

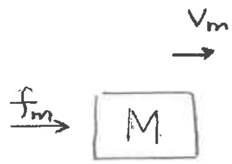
Note: When $s = j\omega$,

$j\omega C = \omega C e^{j90^\circ}$ $\quad Z_C = \frac{1}{j\omega C} = \frac{1}{\omega C e^{j90^\circ}}$

rectangular form $\omega C \angle 90^\circ$ polar form

$= \frac{1}{\omega C} e^{-j90^\circ}$

Impedance of a Mass



Newton's 2nd law for a mass

$$\boxed{f_m = M \frac{d v_m}{dt}}$$

(time-domain representation)

Note:
Just like
$v_L = L \frac{d i_L}{dt}$
for an inductor!

$\updownarrow$

$$\boxed{\frac{d}{dt} \longleftrightarrow s}$$

we are simply relacing $\frac{d}{dt}$ with s!

$$F_m = M \ (s V_m)$$
$$= (sM) V_m$$

Note:
capital letters are often used for forces & velocities when working in the "impedance domain" (i.e. Laplace s-domain).

like voltage

$$\boxed{Z_M = \frac{F_m}{V_m} = sM}$$

like current

(like inductor sL!)

(impedance or Laplace s-domain representation)

Note:

1) Z_M is "small" when s is "small". (near short; little "resistance" to velocity flow)

2) Z_M is "large" when s is "large". (nearly open; large "resistance" to velocity flow)

3) $\boxed{\text{velocity } v_m \text{ of a (real) mass cannot jump!}}$

Why? Such a jump would result in an infinite (∞) force f_m.

This is NOT possible in the real world!

A "large" f_m is possible, but NOT ∞ f_m!

$\boxed{\text{s should be thought of as representing a "frequency".}}$

Think of (angular) frequency w_0 in a sinusoid:

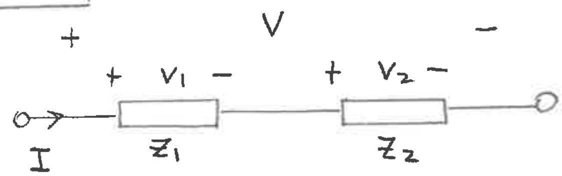
$$A \sin(w_0 t + \theta)$$

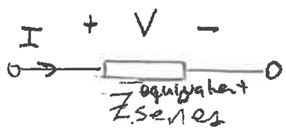
More on this later!

Impedances : In Series & Parallel

900

Series

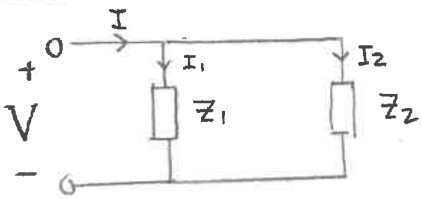


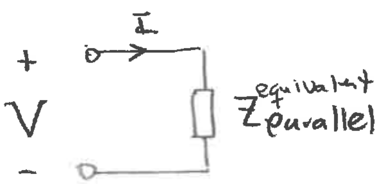


$$V = V_1 + V_2 \quad \text{(KVL)}$$
$$= Z_1 I + Z_2 I \quad \text{(Ohm)}$$
$$= (Z_1 + Z_2) I \quad (I \text{ pass through } Z_1, Z_2)$$
$$= Z_{series}^{equivalent} I$$

$$\boxed{Z_{series}^{equivalent} = Z_1 + Z_2}$$

Parallel





$$I = I_1 + I_2 \quad \text{(KCL)}$$
$$= \left(\frac{V}{Z_1}\right) + \left(\frac{V}{Z_2}\right) \quad \text{(Ohm)}$$
$$= \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) V \quad (V \text{ is across } Z_1 \text{ \& } Z_2)$$
$$= \frac{1}{Z_{parallel}^{equivalent}} V$$

$$V = Z_{parallel}^{equivalent} I$$

$$\frac{1}{Z_{parallel}^{equivalent}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$
$$= \frac{Z_1 + Z_2}{Z_1 Z_2}$$

$$\boxed{Z_{parallel}^{equivalent} = Z_1 \| Z_2 \triangleq \frac{Z_1 Z_2}{Z_1 + Z_2}}$$

1000

Impedance (s-) Domain Methodology

1) Replace all R, L, Cs with their impedances

$$Z_R = R$$
$$Z_L = sL$$
$$Z_C = \frac{1}{sC}$$

2) Apply circuit laws (KVL, KCL, Ohm, etc...) treating all of the impedances (algebraically) as if they were "resistances" Z_i.

3) When all of the algebra is done (see examples that follow), you will be able to read off the system (circuit) differential equation!

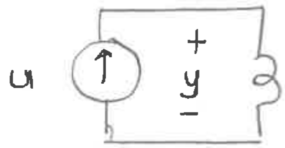
... This is a VERY VERY BIG DEAL FOLKS!

Example 22

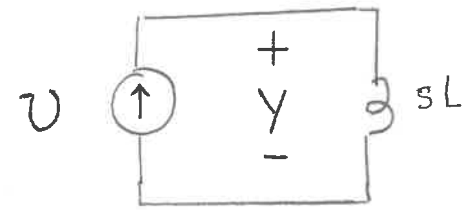
(Introduction to RLC Circuits Using Impedance Concepts)

1010

A Single Inductor



$\Longrightarrow$



$\xRightarrow{\text{ohm}}$ $Y = (sL)U$

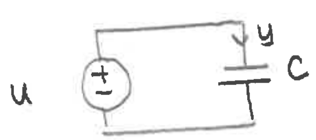
$\Longrightarrow$ $H_{uy} = \frac{Y}{U} = Z_L = sL$

is called the "transfer function" from u to y

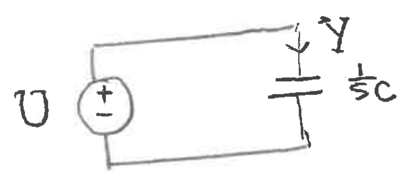
$$\overset{H_{uy}}{\downarrow}$$
$$Y = (sL)U = L(sU) \Longrightarrow y = L\frac{du}{dt}$$

Note: This diff eq can be read off from H_{uy}!

B Single Capacitor



$\Longrightarrow$



$\xRightarrow{\text{ohm}}$ $U = \left(\frac{1}{sC}\right)Y$

$\Longrightarrow$ $Y = (sC)U$

$\Longrightarrow$ $H_{uy} = \frac{Y}{U} = \frac{1}{Z_C} = sC$

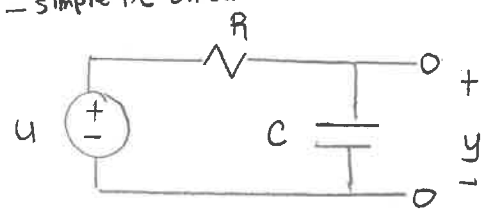
is called the "transfer function" from u to y.

$$\overset{H_{uy}}{\downarrow}$$
$$Y = (sC)U = C(sU) \Longrightarrow y = C\frac{du}{dt}$$

Note: This diff eq can be read off from H_{uy}!

Example 22

$\boxed{C}$ — simple RC circuit

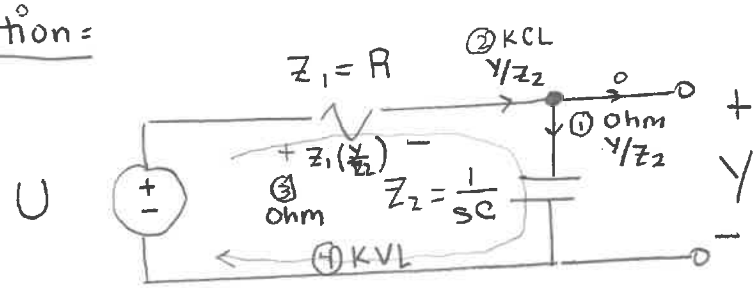


(Simple RC circuit)

3 New Questions:

1. Relate y to u (in the s-domain
2. Find transfer function H_{uy} from u to y
3. Find differential equation relating u to y.

solution:



④ KVL: $U = Z_1\left(\frac{Y}{Z_2}\right) + Y$

$$= \left(\frac{Z_1}{Z_2} + 1\right) Y$$

$$= \left(\frac{Z_1 + Z_2}{Z_2}\right) Y$$

$$\Rightarrow \quad Y = \left(\frac{Z_2}{Z_1 + Z_2}\right) U$$

$$= \left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right) U$$

$$= \left[\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right] U$$

← This is the desired relationship between y & u (in the s-domain)

Note: I multiplied top & bottom by $\frac{s}{R}$ so that this leading coefficient in the denominator is 1!

$$\Rightarrow \quad H_{uy} = \left[\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right]$$ is the transfer function from u to y

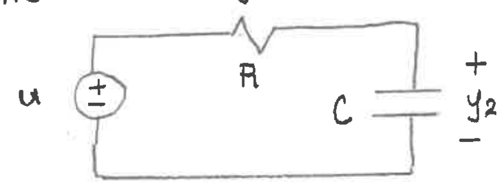
Since $H_{uy} = \frac{Y}{U}$ $\Rightarrow$ $(s + \frac{1}{RC})Y = \frac{1}{RC} U \Rightarrow sY + \frac{1}{RC}Y = \frac{1}{RC}U$

$\Rightarrow \quad \dot{y} + \frac{1}{RC} y = \frac{1}{RC} u$ is the diff eq relating y to u! (1st order diff eq!)

Example 22

(simple RC circuit) $+\ y_1\ -$

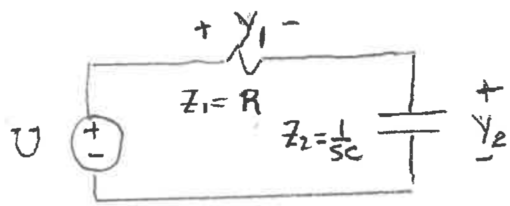
D



← 1 energy storage element C

$\Rightarrow$ 1st order system

1 Relate $y_{1,2}$ to u in the s-domain



By voltage division:

$$Y_1 = \left(\frac{Z_1}{Z_1+Z_2}\right)U = \left(\frac{R}{R+\frac{1}{sC}}\right)U = \left(\frac{s}{s+\frac{1}{RC}}\right)U$$

$$Y_2 = \left(\frac{Z_2}{Z_1+Z_2}\right)U = \left(\frac{\frac{1}{sC}}{R+\frac{1}{sC}}\right)U = \left(\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right)U$$

2 Find transfer functions H_R & H_C (from u to y_1 & from u to y_2)

$$H_R = \frac{s}{s+\frac{1}{RC}} \qquad H_C = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Note: $H_R + H_C = \frac{Y_1}{U} + \frac{Y_2}{U} = \frac{Y_1+Y_2}{U} = \frac{U}{U} = 1$ (by KVL)

3 Find the differential eqs that relate y_1 to u & y_2 to u

$$H_R = \frac{Y_1}{U} = \frac{s}{s+\frac{1}{RC}} \qquad H_C = \frac{Y_2}{U} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

$$\left(s+\frac{1}{RC}\right)Y_1 = sU \qquad \left(s+\frac{1}{RC}\right)Y_2 = \frac{1}{RC}U$$

$$sY_1 + \frac{1}{RC}Y_1 = sU \qquad sY_2 + \frac{1}{RC}Y_2 = \frac{1}{RC}U$$

$$\Rightarrow \quad \dot{y}_1 + \frac{1}{RC}y_1 = \dot{u} \qquad \Rightarrow \quad \dot{y}_2 + \frac{1}{RC}y_2 = \frac{1}{RC}u$$

Note: These are 1st order diff eqs (Why? Because we have only 1 energy storage element C!)

Example 22

D – (simple RC circuit)

4 Determine the characteristic equation $\Phi(s)$ of the system.

equals by definition

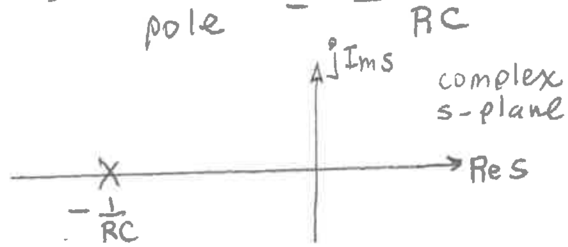
$$\boxed{\Phi(s) \triangleq \text{denominator of any system transfer function}}$$

$= s + \frac{1}{RC}$. This is sometimes referred to as the characteristic polynomial of the system!

5 Determine the poles (or characteristic roots) of the system.

> The poles (or characteristic roots) of a system are found by finding the roots of the system's characteristic equation (polynomial): $\Phi(s) = 0$

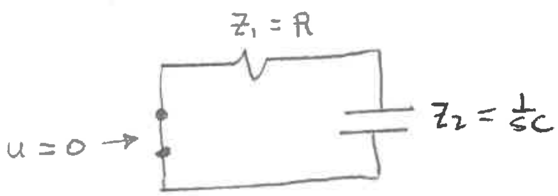
$\Phi(s) = s + \frac{1}{RC} = 0 \implies$ system pole $= -\frac{1}{RC}$



Note: Our system has only 1 pole because it only has 1 energy storage element (C) ...
... it's a 1st order system!

Here is another way to find the poles of the system.

Find the impedance (after setting all inputs to zero).



Note: This was found without any KVL, KCL, or Ohm!

$Z = Z_1 + Z_2$

$= R + \frac{1}{sC} = \frac{RCs + 1}{sC} = R\left[\frac{s + \frac{1}{RC}}{s}\right]$

Now solve $Z = 0 \implies s = -\frac{1}{RC}$ (as obtained earlier!)

Example 22

$\boxed{0}$ — (simple RC circuit)

$\boxed{6}$ Is the system stable, marginally stable, or unstable?

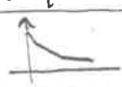
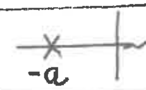
if and only if

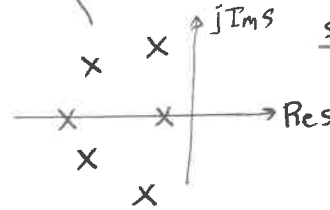
System is stable $\overset{\text{iff}}{\Longleftrightarrow}$ All system poles lie in open left half plane (Re pole < 0)

Remember!!

Reason:

(we'll see that) → Such poles are associated with decaying exponentials in time domain!!!


$e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$


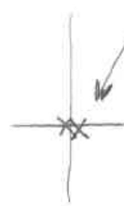


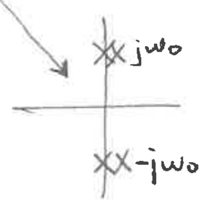
Note: These are "Laplace Transform pairs!" (will learn later!!!)

System is unstable $\overset{\text{iff}}{\Longleftrightarrow}$ $\boxed{1}$ One or more system poles lie in open right half plane (Re pole > 0)

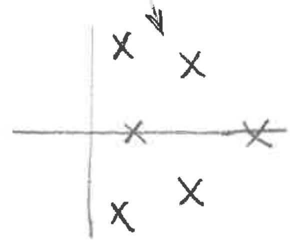
and/or

$\boxed{2}$ System has repeated poles on the imaginary axis

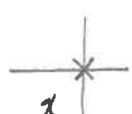
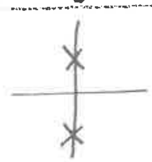

repeated pole at $s=0$



repeated pole at $s=j\omega_0$



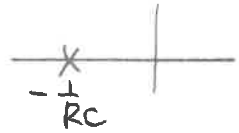
System is marginally stable $\overset{\text{iff}}{\Longleftrightarrow}$ All system poles lie in open left half plane (Re s < 0) except for single (non-repeated) poles on the imaginary axis

single pole at origin

Example 22

D — (simple RC Circuit)



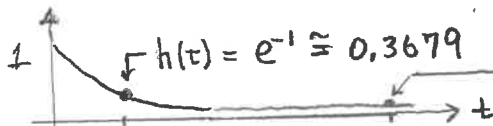
For our system, pole $= -\frac{1}{RC} < 0$

$\Longrightarrow$ System is stable

7 What is the associated (1% or 5 time constant) settling time ?

The following are associated with one another (they are Laplace transform pairs) :

$$h(t) = e^{-at} = e^{-t/\tau} \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s+a} = \frac{1}{s+\frac{1}{\tau}}$$



$h(\tau) = e^{-1} \cong 0.3679$

$h(5\tau) = e^{-5} \cong 0.00674 \cong 0.01$ (decay to 1% !!!)

$\tau \triangleq \frac{1}{a}$ $\qquad t_s \triangleq 5\tau = \frac{5}{a}$

time constant associated with decaying exponential $e^{-at} = e^{-t/\tau}$

For our system, $\quad a = \frac{1}{RC} = \frac{1}{\tau}$

$\Longrightarrow \tau = RC$

$\Longrightarrow t_s \triangleq 5\tau = \frac{5}{a} = 5RC$

Note : When we have complex poles $-\sigma_0 \pm j\omega_0$ the associated 1% or 5 time constant settling time is

$$t_s = \frac{5}{|\mathrm{Re\ poles}|} = \frac{5}{|\mathrm{Re}(-\sigma_0 \pm j\omega_0)|} = \frac{5}{|-\sigma_0|} = \frac{5}{\sigma_0}$$

Example 22

D (Simple RC circuit)

8 Consider a stable system H.

Suppose $u(t) = A + B \underset{\cos}{\sin}(\omega_0 t + \theta)$.

Determine $y_{ss} = \lim_{t \to \infty} y(t)$.

i.e. steady state output = output after a sufficiently long time so that the transients have run their course; i.e. decaying exponentials have effectively reached zero ---- ... $t \geq t_s$!

t_s: system settling time

Answer:

For a stable system H,

if $u(t) = A + B \underset{\cos}{\sin}(\omega_0 t + \theta)$

then $y_{ss} = A\, H(0) + B\,|H(j\omega_0)| \underset{\cos}{\sin}(\omega_0 t + \theta + \angle H(j\omega_0))$

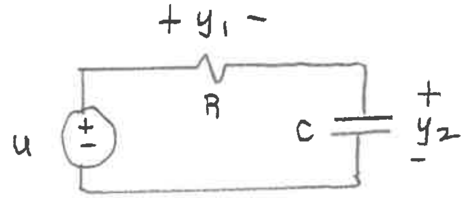
Remember! (FOR EVER!)

This result shall be referred to as the Method of the Transfer Function (MOTF).

Example 22 1080

D — (simple RC circuit)

For our system



we have

$$H_1 = H_R = \frac{Y_1}{U} = \frac{s}{s + \frac{1}{RC}} \qquad H_2 = H_C = \frac{Y_2}{U} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

These transfer functions are stable with poles at $s = -\frac{1}{RC}$

& $t_s = 5\tau = 5RC$

Using the MOTF, we have the following:

if $u(t) = A + B \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta)$

then

$$y_{1ss} = A\, H_1(0) + B\, |H_1(j\omega_0)| \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta + \angle H_1(j\omega_0))$$

$$y_{2ss} = A\, H_2(0) + B\, |H_2(j\omega_0)| \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta + \angle H_2(j\omega_0))$$

Make sure that you write this on an exam.

dc gains of $H_{1,2}$

Magnitude Response of $H_{1,2}$

Phase Response of $H_{1,2}$

Note:

You can write this before (& without computing H_1 & H_2)!

Example 22

1090

$\boxed{D}$ – (simple RC circuit)

For our system

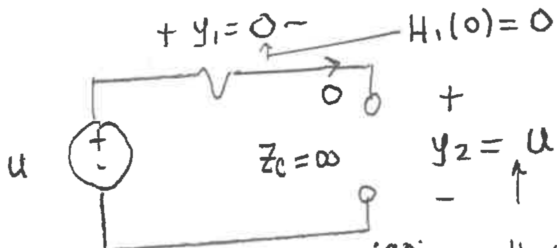
$$H_1 = \frac{s}{s + \frac{1}{RC}} \qquad H_2 = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Hence, we have dc gains:

$$H_1(0) = 0 \qquad H_2(0) = 1$$

<u>Why?</u> At dc $(s=0)$, $Z_C = \frac{1}{sC}\Big|_{s=0} = \infty$ (cap is open! & no current flows)

& our circuit looks like:



$j\omega = \omega e^{j90^\circ}$ polar form, rectangular form, $\omega \angle 90^\circ$

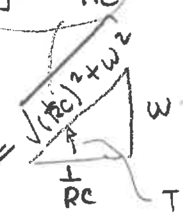
rectangular form, polar form: $\frac{1}{RC} = \frac{1}{RC} e^{j0^\circ}$

We also have

$$H_1(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}} \qquad H_2(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

$$\frac{1}{RC} + j\omega = \sqrt{\left(\tfrac{1}{RC}\right)^2 + \omega^2}\, e^{j\,\mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)}$$

rectangular form, polar form

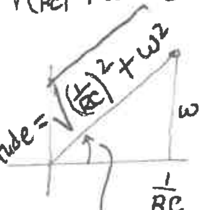


SUMMARY

$j\omega = \omega e^{j90^\circ}$ magnitude ω, angle 90°

$\frac{1}{RC} = \frac{1}{RC} e^{j0^\circ}$ angle $= 0$, magnitude $= \frac{1}{RC}$

$\frac{1}{RC} + j\omega = \sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}\ e^{j\,\mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)}$



Angle $= \mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$

$$|H_1(j\omega)| = \frac{\text{mag of top}}{\text{mag of bottom}} = \frac{\omega}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}$$

$$|H_2(j\omega)| = \frac{\text{mag of top}}{\text{mag of bottom}} = \frac{\frac{1}{RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}$$

$$\angle H_1(j\omega) = \angle\text{top} - \angle\text{bottom} = 90^\circ - \mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$$

$$\angle H_2(j\omega) = \angle\text{top} - \angle\text{bottom} = 0 - \mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right) = -\mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$$

Example 22

D — (simple RC circuit)

For our system

$$H_1 = \frac{s}{s + \frac{1}{RC}} \qquad H_2 = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$H_1(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}} \qquad H_2(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Magnitude & Phase Response for H_1

$$|H_1(j\omega)| = \frac{\omega}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}$$

$$\angle H_1(j\omega) = 90^\circ - \operatorname{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$$

Magnitude & Phase Response for H_2

$$|H_2(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}$$

$$\angle H_2(j\omega) = -\operatorname{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$$

If $\quad u(t) = A + B \begin{matrix} \sin \\ \cos \end{matrix} (\omega_0 t + \theta)$

sin or cos

by the MOTF

$$y_{1ss} = H_1(0)\,A + B\,|H_1(j\omega_0)| \begin{matrix} \sin \\ \cos \end{matrix} \left(\omega_0 t + \theta + \angle H_1(j\omega_0)\right)$$

$$= (0)A + B \frac{\omega_0}{\sqrt{\omega_0^2 + \left(\frac{1}{RC}\right)^2}} \begin{matrix} \sin \\ \cos \end{matrix} \left(\omega_0 t + \theta + 90^\circ - \operatorname{Tan}^{-1}\left(\frac{\omega_0}{\frac{1}{RC}}\right)\right)$$

$$y_{2ss} = H_2(0)\,A + B\,|H_2(j\omega_0)| \begin{matrix} \sin \\ \cos \end{matrix} \left(\omega_0 t + \theta + \angle H_2(j\omega_0)\right)$$

$$= (1)\,A + B \frac{\frac{1}{RC}}{\sqrt{\omega_0^2 + \left(\frac{1}{RC}\right)^2}} \begin{matrix} \sin \\ \cos \end{matrix} \left(\omega_0 t + \theta - \operatorname{Tan}^{-1}\left(\frac{\omega_0}{\frac{1}{RC}}\right)\right)$$

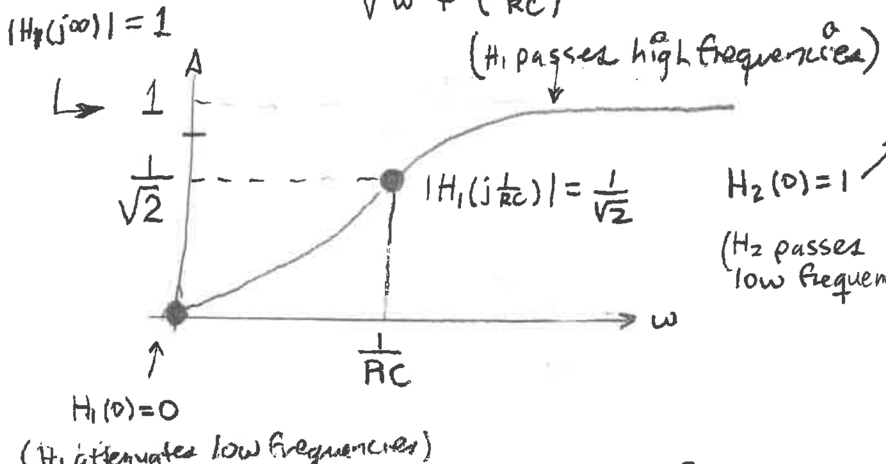
<u>Note</u>: On an exam, it is not necessary to make the above substitutions --- computing the essential quantities suffices!

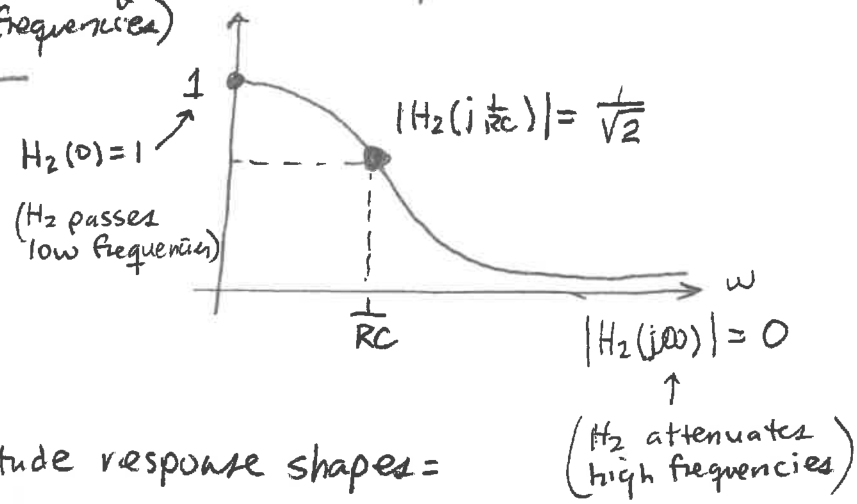
Example 22

D — (simple RC circuit)

It is useful, & highly instructive, to plot $\left\{ \begin{array}{l} |H_1(j\omega)|, |H_2(j\omega)| \\ \underline{/H_1(j\omega)}, \underline{/H_2(j\omega)} \\ \text{vs } \omega \text{ !!!!} \end{array} \right\}$

$$|H_1(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}} \qquad |H_2(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$$



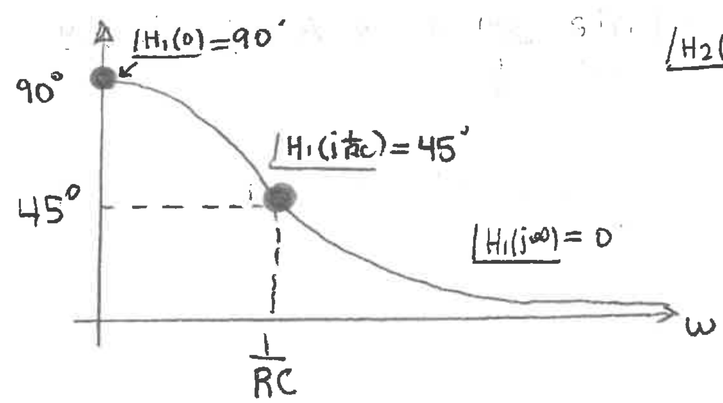


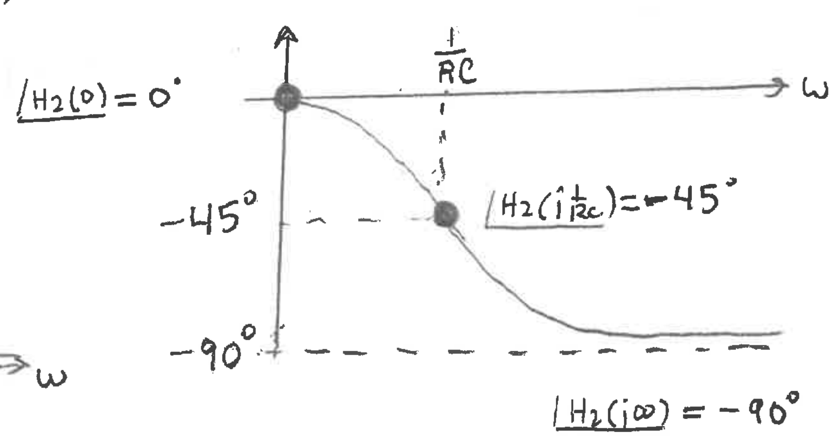
Because of the above magnitude response shapes:

H_1 is called a High Pass Filter (HPF)	H_2 is called a Low Pass Filter (LPF)

We also have

$$\underline{/H_1(j\omega)} = 90^\circ - \mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right) \qquad \underline{/H_2(j\omega)} = -\mathrm{Tan}^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)$$





D

Example 22

(simple RC circuit)

The following is a 3 frequency summary for

$$u(t) = A + B \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta)$$

ω_0 small ($<< \frac{1}{RC}$) $\qquad H_1 \approx \frac{s}{\frac{1}{RC}} \qquad H_2 \approx 1$

$$y_{1ss} \cong (0)A + B\left(\frac{\omega_0}{\frac{1}{RC}}\right) \begin{matrix}\sin \\ \cos\end{matrix} \left(\omega_0 t + \theta + 90^\circ - \cancelto{\sim 0^\circ}{\tan^{-1}\left(\frac{\omega_0}{\frac{1}{RC}}\right)}\right)$$

$$y_{2ss} \cong (1)A + B(1) \begin{matrix}\sin \\ \cos\end{matrix} \left(\omega_0 t + \theta - \cancelto{\sim 0^\circ}{\tan^{-1}\left(\frac{\omega_0}{\frac{1}{RC}}\right)}\right)$$

$\omega_0 = \frac{1}{RC}$

$$y_{1ss} \cong (0)A + B\frac{1}{\sqrt{2}} \begin{matrix}\sin \\ \cos\end{matrix} \left(\omega_0 t + \theta + 90^\circ - \cancelto{45^\circ}{\tan^{-1}(1)}\right)$$

$$y_{2ss} \cong (1)A + B\frac{1}{\sqrt{2}} \begin{matrix}\sin \\ \cos\end{matrix} \left(\omega_0 t + \theta - \cancelto{45^\circ}{\tan^{-1}(1)}\right)$$

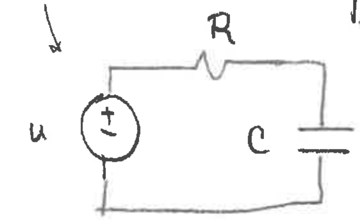
ω_0 large ($>> \frac{1}{RC}$) $\qquad H_1 \cong 1e^{j0^\circ} \qquad H_2 \cong \frac{\frac{1}{RC}}{s}$

$$y_{1ss} \cong (0)A + B(1) \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta + 0^\circ)$$

$$y_{2ss} \cong (1)A + B\frac{\frac{1}{RC}}{\omega_0} \begin{matrix}\sin \\ \cos\end{matrix} (\omega_0 t + \theta - 90^\circ)$$

Example 22

(Simple RC circuit)

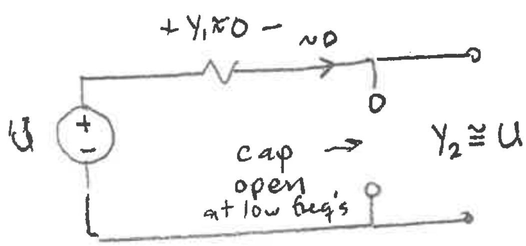


D (simple RC circuit)

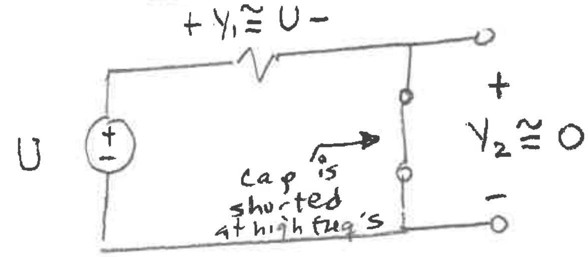
Here is how our circuit looks for a sinusoidal input:

$$u(t) = B \,\begin{matrix}\sin \\ \cos\end{matrix}\, (\omega_0 t + \theta)$$

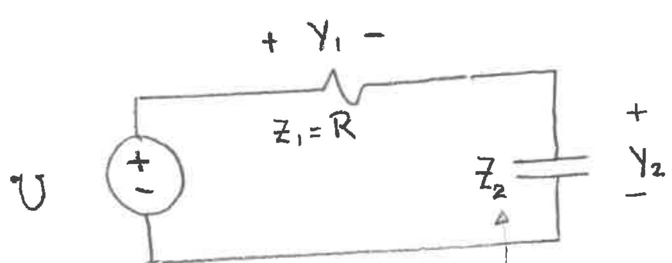
$\boxed{\omega_0 \cong 0}$ (Cap ~ open)



$\boxed{\omega_0 \cong \infty}$ (Cap ~ shorted)



$\boxed{\omega_0 = \frac{1}{RC}}$



$$Z_2 = \frac{1}{j\omega_0 C} = \frac{1}{j\left(\frac{1}{RC}\right)C} = \left[\frac{1}{j\left(\frac{1}{R}\right)}\right]\left[\frac{x(-j)}{x(-j)}\right]$$

$$= \left(\frac{R}{j}\right)\left(\frac{-j}{-j}\right)$$

$$= \frac{-jR}{-(j)(j)}$$

$$= \frac{-jR}{-(-1)}$$

$$= -jR$$

$$Y_1 = \left(\frac{Z_1}{Z_1+Z_2}\right)U = \left(\frac{R}{R-jR}\right)U = \left(\frac{1}{1-j1}\right)U$$

$$= \frac{1}{\sqrt{2}\,e^{-j45^\circ}}\,U$$

$$= \frac{1}{\sqrt{2}}\,e^{j45^\circ}\,U$$

1, 1, $\sqrt{2}$, $45°$

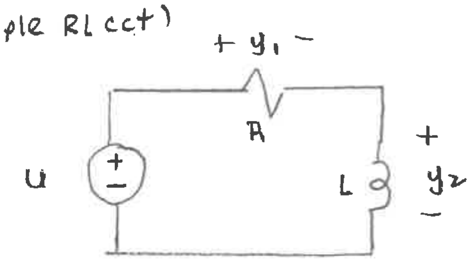
$$Y_2 = \left(\frac{Z_2}{Z_1+Z_2}\right)U = \left(\frac{-jR}{R-jR}\right)U = \left(\frac{-j1}{1-j1}\right)U = \frac{1e^{-j90^\circ}}{\sqrt{2}\,e^{-j45^\circ}}\,U$$

$$= \frac{1}{\sqrt{2}}\,e^{-j45^\circ}\,U$$

Example 22

1140

E — (simple RL cct)



(Simple RL Circuit)

1 Relate $y_{1,2}$ to u (in s-domain)

$$Y_1 = \left(\frac{Z_1}{Z_1+Z_2}\right)U = \left(\frac{R}{R+sL}\right)U = \left[\frac{\frac{R}{L}}{s+\frac{R}{L}}\right]U$$

$$Y_2 = \left(\frac{Z_2}{Z_1+Z_2}\right)U = \left(\frac{sL}{R+sL}\right)U = \left[\frac{s}{s+\frac{R}{L}}\right]U$$

2 Find transfer functions $H_{1,2}$ from u to $y_{1,2}$

$$H_1 = \frac{\frac{R}{L}}{s+\frac{R}{L}}$$

$$H_2 = \frac{s}{s+\frac{R}{L}}$$

3 Find diff eqs relating $y_{1,2}$ to u

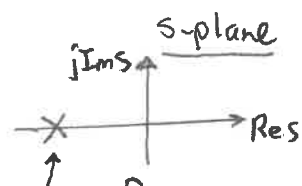
$$\dot{y}_1 + \frac{R}{L}y_1 = \left(\frac{R}{L}\right)u$$

$$\dot{y}_2 + \frac{R}{L}y_2 = \dot{u}$$

4 Find characteristic eq (polynomial) of system

$$\Phi(s) = \text{denominator of } H_1 \text{ or } H_2 = s + \frac{R}{L}$$

5 Determine pole (characteristic root) of system.



Solve $\Phi(s) = 0 \implies \Phi(s) = s + \frac{R}{L} = 0 \implies s = -\frac{R}{L}$

6 Determine 1% (or 5 time constant) settling time t_s of system

$$t_s = 5\tau = \frac{5}{|\text{Re pole}|} = \frac{5}{\left|-\frac{R}{L}\right|} = 5\frac{L}{R}$$

7 Is system stable, marginally stable, or unstable?

System is stable iff all poles lie in open LHP (Re pole < 0) $\implies$ Our system has pole at $s = -\frac{R}{L}$ (in LHP)

$\implies$ stable

Example 22

simple RL cct

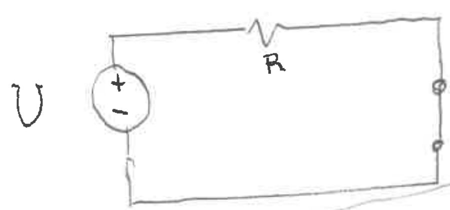
E **8** Find $y_{1,2ss}$ when (for simple RL circuit) $u = A + B \sin(\omega_0 t + \theta)$ cos

MOTF $\Longrightarrow$ $y_{iss} = H_i(0)\,A + B\,|H_i(j\omega_0)| \sin(\omega_0 t + \theta + \angle H_i(j\omega_0))$ Cos

Since $H_1(s) = \dfrac{\frac{R}{L}}{s + \frac{R}{L}}$ (H_R) & $H_2(s) = \dfrac{s}{s + \frac{R}{L}}$ (H_L)

$H_1(0) = 1$ $\qquad$ $H_2(0) = 0$ (since inductor looks like a short at dc & all the voltage U appears across R !)

+ $Y_1 = U$ −

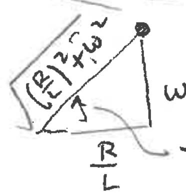


+ $Y_2 = 0$ −

$\frac{R}{L} = \frac{R}{L} e^{j0°}$ $\qquad$ $j\omega = \omega e^{j90°}$

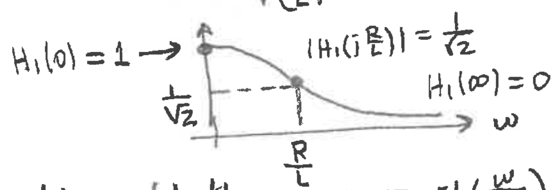
$H_1(j\omega) = \dfrac{\frac{R}{L}}{j\omega + \frac{R}{L}}$ $\qquad$ $H_2(j\omega) = \dfrac{j\omega}{j\omega + \frac{R}{L}}$

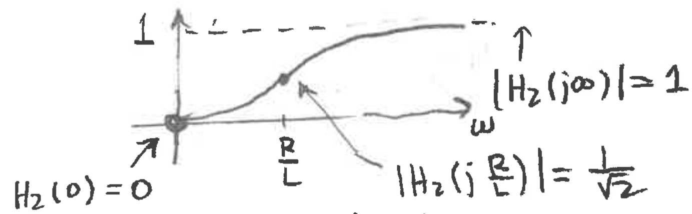
$\frac{R}{L} + j\omega = \sqrt{(\frac{R}{L})^2 + \omega^2}\; e^{j \mathrm{Tan}^{-1}\left(\frac{\omega}{R/L}\right)}$



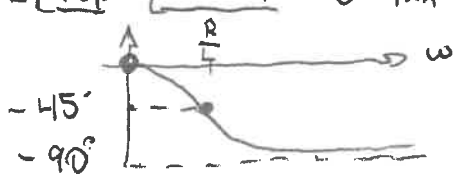
$\mathrm{Tan}^{-1}\left(\frac{\omega}{R/L}\right)$

$|H_1(j\omega)| = \dfrac{|\text{top}|}{|\text{bottom}|} = \dfrac{\frac{R}{L}}{\sqrt{(\frac{R}{L})^2 + \omega^2}}$ $\qquad$ $|H_2(j\omega)| = \dfrac{|\text{top}|}{|\text{bottom}|} = \dfrac{\omega}{\sqrt{(\frac{R}{L})^2 + \omega^2}}$

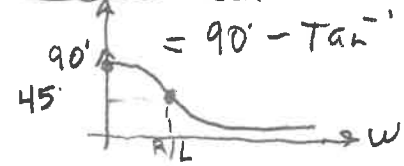




$\angle H_1(j\omega) = \angle\text{top} - \angle\text{bottom} = 0 - \mathrm{Tan}^{-1}\left(\frac{\omega}{R/L}\right)$

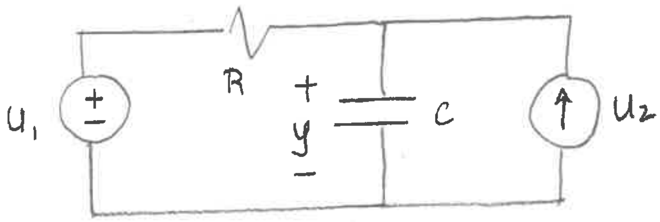


$\angle H_2(j\omega) = \angle\text{top} - \angle\text{bottom}$
$= 90° - \mathrm{Tan}^{-1}\left(\frac{\omega}{R/L}\right)$

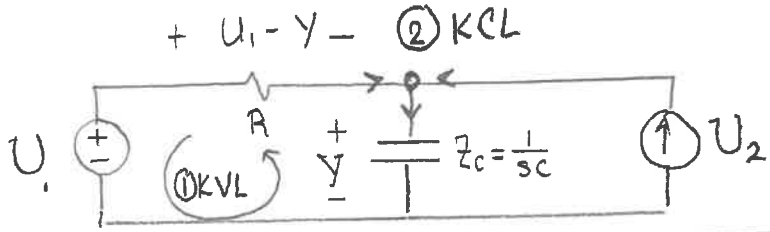


Example 22

F — (RC cct with 2 sources)



1 Relate y to $u_{1,2}$ (in s-domain)



② KCL (+Ohm) =

$$\boxed{\left(\frac{U_1 - Y}{R}\right) + \left(U_2\right) = \left(\frac{Y}{Z_C}\right) = sCY}$$

$$\xrightarrow{\text{algebra}} \quad Y\left[sC + \frac{1}{R}\right] = \left[\frac{1}{R_1}\right]u_1 + [1]\,U_2$$

$$\Longrightarrow \quad Y\left[s + \frac{1}{RC}\right] = \left[\frac{1}{RC}\right]u_1 + \left[\frac{1}{C}\right]u_2$$

$$\Longrightarrow \quad Y = \left[\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right]u_1 + \left[\frac{\frac{1}{C}}{s + \frac{1}{RC}}\right]U_2$$

2 Find transfer functions $H_{1,2}$ from u to y.

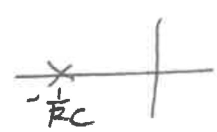
$$H_1 = \frac{Y}{U_1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \qquad H_2 = \frac{Y}{U_2} = \frac{\frac{1}{C}}{s + \frac{1}{RC}}$$

3 Find diff eq relating y to $u_{1,2}$. $\quad \dot{y} + \frac{1}{RC}y = \frac{1}{RC}u_1 + \frac{1}{C}u_2$

4 Characterist eq (poly) = $\quad \Phi(s) =$ denominator of H_1 or H_2

$$= s + \frac{1}{RC}$$

5 Poles (Characteristic roots) = $\quad \Phi(s) = s + \frac{1}{RC} = 0 \Longrightarrow s = -\frac{1}{RC}$



6 Stability = All poles lie in LHP $\Longrightarrow$ System is stable

7 t_s: $\quad t_s = \frac{5}{|\text{Re pole}|} = \frac{5}{|\text{Re}(-\frac{1}{RC})|} = 5RC$

Example 22

F — (RC cct with 2 sources)

B Find y_{ss} when $u_1 = A_1 + B_1 \sin(\omega_1 t + \theta_1)$ & $u_2(t) = A_2 + B_2 \sin(\omega_2 t + \theta_2)$

MOTF $\Longrightarrow$

$$y_{ss} = y_{1ss} + y_{2ss}$$

$$y_{iss} = H_i(0) A_i + B_i \, |H_i(j\omega_i)| \, \underset{\cos}{\sin}\left(\omega_i t + \theta_i + \angle H_i(j\omega_i)\right)$$

$$H_1(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \qquad H_1(0) = 1$$

$$H_1(j\omega_1) = \frac{\frac{1}{RC}}{j\omega_1 + \frac{1}{RC}} \Longrightarrow \quad |H_1(j\omega_1)| = \frac{\frac{1}{RC}}{\sqrt{\omega_1^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\angle H_1(j\omega_1) = -\mathrm{Tan}^{-1}\left(\frac{\omega_1}{\frac{1}{RC}}\right)$$

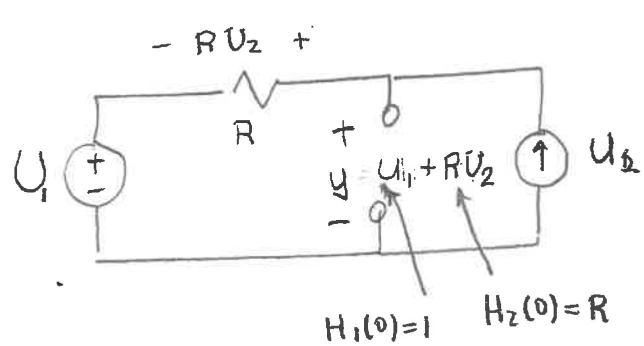
$$H_2(s) = \frac{\frac{1}{C}}{s + \frac{1}{RC}} \qquad H_2(0) = R$$

$$H_2(j\omega_2) = \frac{\frac{1}{C}}{j\omega_2 + \frac{1}{RC}} \Longrightarrow \quad |H_2(j\omega_2)| = \frac{\frac{1}{C}}{\sqrt{\omega_2^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\angle H_2(j\omega_2) = -\mathrm{Tan}^{-1}\left(\frac{\omega_2}{\frac{1}{RC}}\right)$$

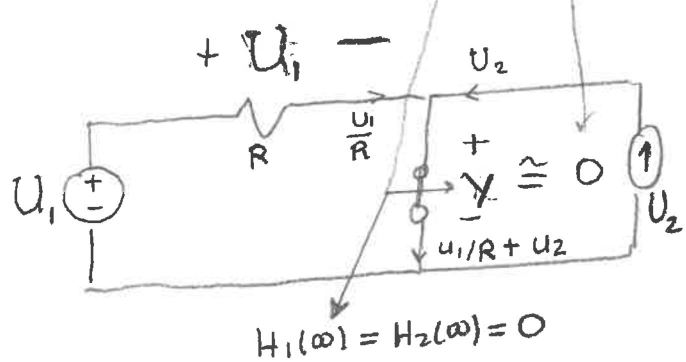
Note:

At dc (low freq) circuit looks like =



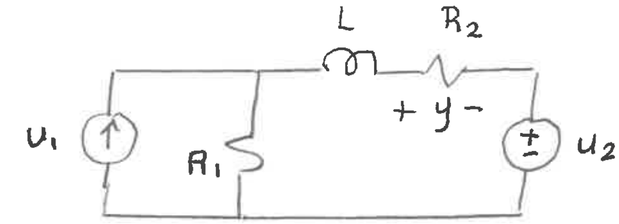
(cap looks like an open cct at low freq's & dc)

At high freq's circuit looks like = $0U_1 + 0H_2$



(cap looks like a short cct at high freq's)

Problem 22 | 1180



1. Determine the system pole, time constant $\tau = \frac{1}{|\text{pole}|}$, & (1% or 5 time constant) settling time $t_s = 5\tau = \frac{5}{|\text{pole}|}$ without any KVL, KCL, Ohm!

Hint: Set $u_1 = 0$ (open)
$u_2 = 0$ (short)

Compute impedance =

$Z = R_1 + sL + R_2$

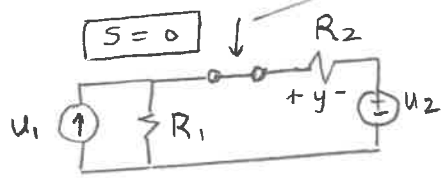
& see when $Z = 0$.

2. Determine $H_1(0), H_2(0)$ & $H_1(\infty), H_2(\infty)$ by shorting or opening L

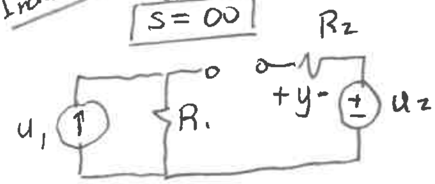
Hint: Inductor $Z_L = sL$ looks like a short at $s=0$

Inductor $Z_L = sL$ looks like an open at $s = \infty$

$\boxed{s = 0}$



$\boxed{s = \infty}$



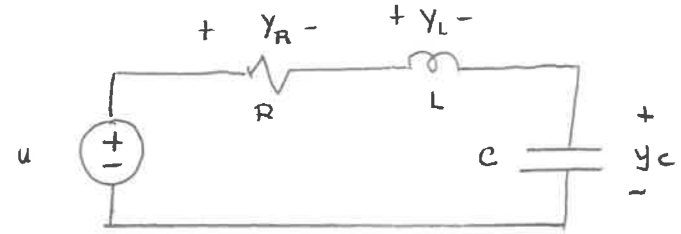
3. Relate y to $u_{1,2}$ (in - s domain)
4. Determine transfer functions H_1, H_2 from u_1 to y & from u_2 to y.
5. Determine diff eq relating y to u_1, u_2.
6. Determine characteristic polynomial $\Phi(s)$.
7. Determine pole of system (characteristic root).
8. Is system stable, marginally stable, or unstable?
9. Determine system time constant τ & settling time t_s.
10. For $R_1 = R_2 = L = 1$, determine y_{ss} when $u_1 = 10 + \cos(2t + 45°)$
$u_2 = 100 \sin(200t + 90°)$

Hint: $-\frac{1}{j200+2} \cong -\frac{1}{200} e^{-j90°} = \frac{1}{200} e^{j(180-90)}$

Example 23

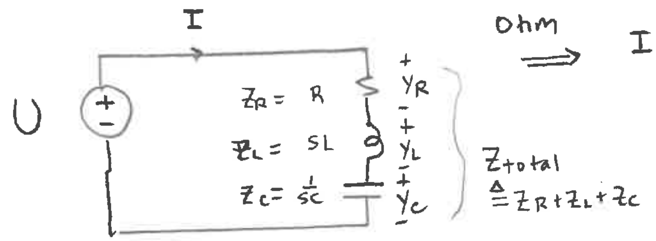
Intro. to (RLC Circuits)

1190



$\leftarrow$ 2 energy storage elements: L, C

1 Relate $y_{R,L,C}$ to u (in s-domain)



$$\xrightarrow{\text{Ohm}} \quad I = \frac{U}{Z_{total}} = \frac{U}{Z_R + Z_L + Z_C}$$

$$= \frac{U}{R + sL + \frac{1}{sC}}$$

(Multiply top & bottom by s/L)

$$= \left[\frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right] U$$

Note: $H_{UI} = \dfrac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$

$\uparrow$ Transfer function from u to i

$$\Rightarrow y_R = (R) I = \left[\frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right] U$$

$$y_L = (sL) I = \left[\frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right] U$$

$$y_C = \left(\frac{1}{sC}\right) I = \left[\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right] U$$

2 Find transfer functions from u to $y_{R,L,C}$.

$$H_R = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H_L = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H_C = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Note:

$$H_R + H_L + H_C = \frac{y_R}{U} + \frac{y_L}{U} + \frac{y_C}{U}$$

$$= \frac{U}{U} \quad \text{(by KVL)}$$

$$= 1$$

Example 23

[3] Find diff eqs relating $y_{R,L,C}$ to u

$$H_R = \frac{y_R}{U} = \frac{\frac{R}{L}s}{s^2+\frac{R}{L}s+\frac{1}{LC}} \Longrightarrow \ddot{y}_R + \frac{R}{L}\dot{y}_R + \frac{1}{LC}y_R = \frac{R}{L}\dot{u}$$

$$H_L = \frac{y_L}{U} = \frac{s^2}{s^2+\frac{R}{L}s+\frac{1}{LC}} \Longrightarrow \ddot{y}_L + \frac{R}{L}\dot{y}_L + \frac{1}{LC}y_L = \ddot{u}$$

$$H_C = \frac{y_C}{U} = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}} \Longrightarrow \ddot{y}_C + \frac{R}{L}\dot{y}_C + \frac{1}{LC}y_C = \frac{1}{LC}u$$

[4] Char eq (poly) = $\Phi(s)$ = denominator of any system tf

$$= s^2 + \frac{R}{L}s + \frac{1}{LC}$$

[5] System Poles (Characteristic Roots) =

$$\Phi(s) = s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\boxed{\begin{array}{c} as^2+bs+c=0 \\ \text{Quadratic} \Longrightarrow \text{Formula} \quad s_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \end{array}}$$

$$= \frac{-\left(\frac{R}{L}\right) \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2(1)}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Lets try to use some basics from quadratics...

$$\boxed{\text{Discriminant} \triangleq b^2-4ac}$$

$$= \left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)$$

Example 23

For what critical value of resistance R_c, is discriminant zero?

$$\text{Discriminant} = \left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right) = 0$$

$$\Leftrightarrow \left(\frac{R}{L}\right)^2 = \frac{4}{LC}$$

$$\Leftrightarrow \frac{R}{L} = \frac{2}{\sqrt{LC}}$$

$$\Leftrightarrow R = \frac{2L}{\sqrt{LC}}$$

$$\text{Discriminant} = 0 \Leftrightarrow R = \boxed{R_c = 2\sqrt{\frac{L}{C}}}$$

From this, we know that there are 3 cases of interest:

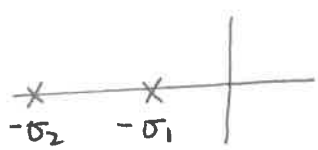
Overdamped	Discriminant > 0	$(R > R_c)$ "resistance large"	Roots Real, Stable, Distinct
Critically Damped	" $= 0$	$(R = R_c)$	Roots Real, Stable, Repeated
Underdamped	" < 0	$(0 < R < R_c)$ "resistance small"	Roots Complex, Stable, Conjugate
		For $R = 0$,	Roots purely Imaginary, marginally stable, conjugate

Overdamped

$(\text{Disc} > 0,\ R > R_c)$ – "large" resistance

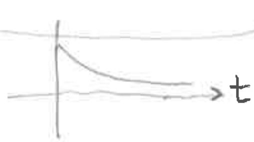
$$\underset{\text{(poles)}}{\text{Roots}} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}},\ -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\sigma_1,\ -\sigma_2$$



roots are real stable distinct

Fact: We'll see that (unforced) time solutions look like decaying exponentials $A_1 e^{-\sigma_1 t}, A_2 e^{-\sigma_2 t}$



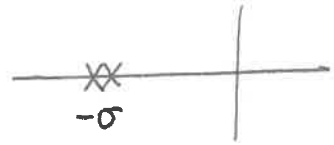
$$t_s = 5\tau = \frac{5}{|\text{the slowest pole}|} = \frac{5}{\sigma_2}$$

Example 23

Critically Damped

$(\text{Disc} = 0,\ R = R_c)$ — "critical" resistance

(Poles)
$$\text{Roots} = -\frac{R}{2L},\ -\frac{R}{2L}$$
$$= -\sigma, -\sigma$$



Fact:
we'll see that (unforced) time solutions look like decaying exponentials
$A_1 t e^{-\sigma t} + A_2 e^{-\sigma t}$

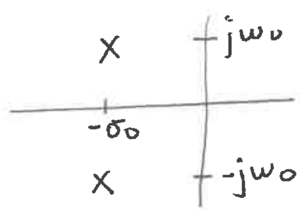
roots are real, stable, repeated

$$t_s = 5\tau = \frac{5}{|\text{Re (slowest pole)}|} = \frac{5}{\sigma}$$

Underdamped

$(\text{Disc} < 0,\ 0 < R < R_c)$ — "small" resistance

(poles)
$$\text{roots} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$
$$= -\sigma_o \pm j\omega_o$$



Fact:
we'll see that (unforced) time solutions look like decaying exponential sinusoids:
$$|A_1| e^{-\sigma_o t} \sin(\omega_o t + \angle A_1)$$

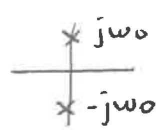
roots are complex, stable, conjugates

$$t_s = 5\tau = \frac{5}{|\text{Re (poles)}|} = \frac{5}{|\text{Re}(-\sigma_o \pm j\omega_o)|} = \frac{5}{|-\sigma_o|} = \frac{5}{\sigma_o}$$

Note:

For $R = 0$ (No resistance),

(poles)
$$\text{roots} = \pm j\frac{1}{\sqrt{LC}}$$



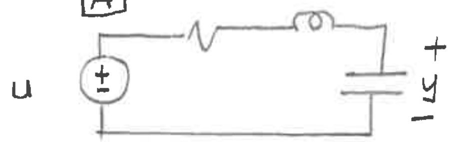
roots are purely imaginary, marginally stable, conjugates

In such a case, we say that the system is <u>Undamped</u>.

Problem 23

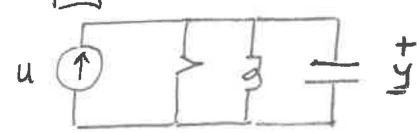
1230

A



(R = L = C = 1)

B



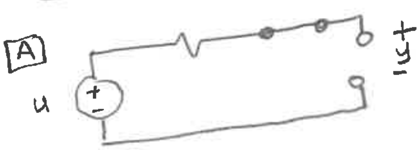
(R = L = C = 1)

For each circuit:

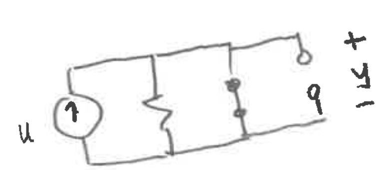
1. Relate y to u (in s-domain)
2. Determine transfer function H from u to y
3. Determine diff eq relating y to u.
4. Determine system characteristic polynomial, poles, time constant τ, & settling time t_s. (Is system stable, marginally stable, or unstable?)
5. Determine y_{ss} when $u(t) = A + B \sin(0.01t - 90°)$
   $- C \cos(t + 45°)$
   $+ D \sin(100t - 30°)$
6. Compute H(0) & H(∞) by shorting or opening L & C.
   Compare what you get here with that using H found in 2 -
   You MUST get the same answer!
   Hint: Use the following

For $s = 0$ we have

A

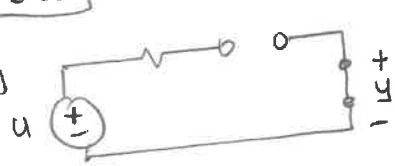


B

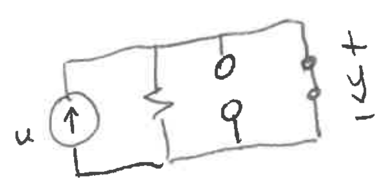


For $s = \infty$ we have

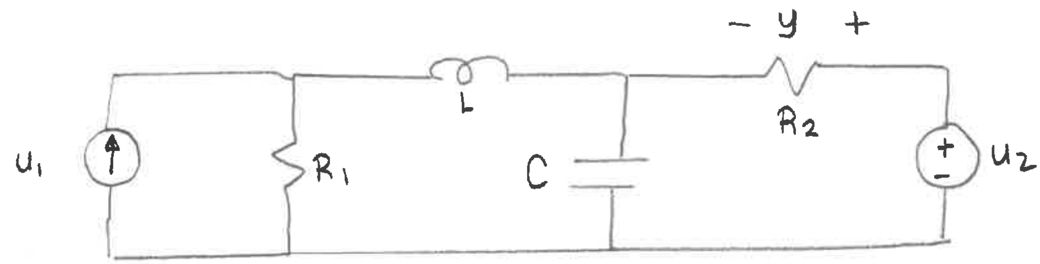
A



B



Example 24 (2nd Order RLC Cct with 2 Sources) 1240

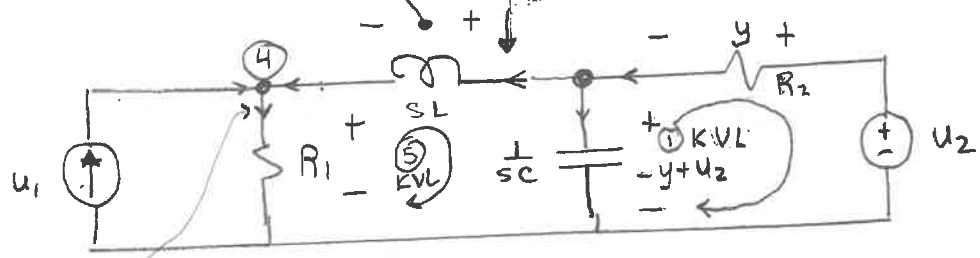


Note: Here, I shall use a straight KVL, KCL, Ohm approach.

1 Relate y to $u_{1,2}$ (in s-domain)

③ Ohm $= sL\left\{\frac{y}{R_2} - sC[-y+u_2]\right\}$

② KCL (+Ohm) $\frac{y}{R_2} - sC[-y+u_2]$



④ KCL $= u_1 + \left\{\frac{y}{R_2} - sC[-y+u_2]\right\}$

⑤ KVL = (+Ohm)

$$\boxed{R_1\left[u_1 + \left\{\frac{y}{R_2} - sC[-y+u_2]\right\}\right] = -sL\left\{\frac{y}{R_2} - sC[-y+u_2]\right\} + [-y+u_2]}$$

$$Y\left[\frac{R_1}{R_2} + R_1 sC + \frac{sL}{R_2} + s^2LC + 1\right] = u_1[-R_1] + u_2[R_1 sC + s^2LC + 1]$$

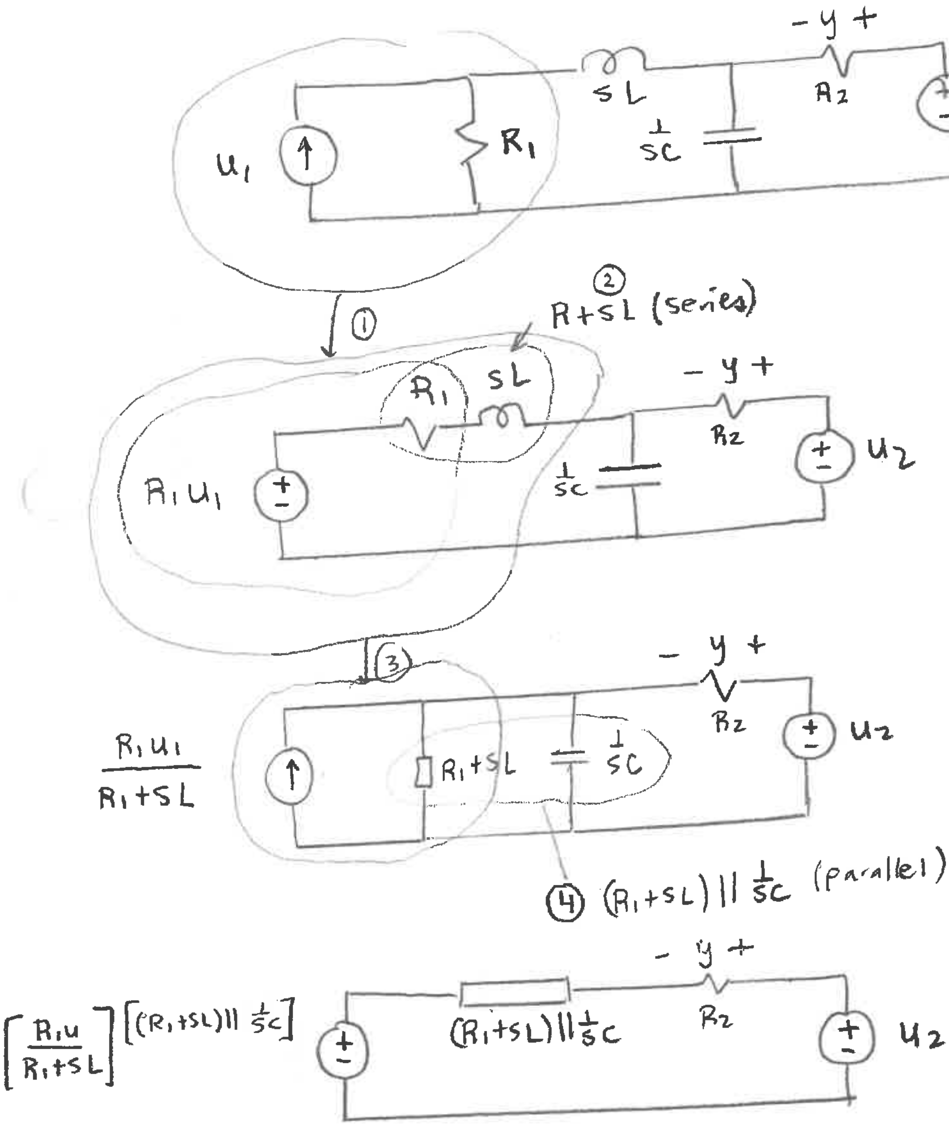
$$LC\; Y\left[\frac{R_1}{R_2LC} + \frac{R_1Cs}{LC} + \frac{sL}{R_2LC} + \frac{s^2LC}{LC} + \frac{1}{LC}\right] = u_1[-R_1] + u_2 LC\left[\frac{R_1C}{LC}s + \frac{s^2LC}{LC} + \frac{1}{LC}\right]$$

$$Y\left[\frac{R_1}{R_2LC} + \frac{R_1}{L}s + \frac{s}{R_2C} + s^2 + \frac{1}{LC}\right] = u_1\left[-\frac{R_1}{LC}\right] + u_2\left[\frac{R_1}{L}s + s^2 + \frac{1}{LC}\right]$$

Example 24

$$Y = \left[\frac{-\frac{R_1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)s + \left(1 + \frac{R_1}{R_2}\right)\frac{1}{LC}} \right] U_1 + \left[\frac{s^2 + \frac{R_1}{L}s + \frac{1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)s + \left(1 + \frac{R_1}{R_2}\right)\frac{1}{LC}} \right] U_2$$

Now lets approach the above using source transformations.



← source transformations have permitted us to show precisely what y "directly sees"!

From this KVL (+ ohm) yields

$$\left[\frac{R_1 u}{R_1 + sL}\right]\left[(R_1+sL) \,\|\, \tfrac{1}{sC}\right] = -\left[(R_1+sL)\,\|\,\tfrac{1}{sC}\right]\frac{y}{R_2} - y + u_2$$

Example 24

After a little algebra, we get:

$$y\left[\frac{(R_1+sL)\,||\,\frac{1}{sC}}{R_2}+1\right]=-\left\{\frac{\left[(R_1+sL)\,||\,\frac{1}{sC}\right]R_1}{R_1+sL}\right\}u_1+u_2$$

or

$$\boxed{y=-\left[\frac{\dfrac{R_1\left[(R_1+sL)\,||\,\frac{1}{sC}\right]}{R_1+sL}}{1+\dfrac{(R_1+sL)\,||\,\frac{1}{sC}}{R_2}}\right]u_1+\left[\frac{1}{1+\dfrac{(R_1+sL)\,||\,\frac{1}{sC}}{R_2}}\right]U_2}$$

This is the cleanest form of the relationship between y, u_1, u_2.... because it is fundamentally based on what y "directly sees!"

Lets do some algebra ---

$$(R_1+sL)\,||\,\frac{1}{sC}=\frac{\dfrac{R_1+sL}{sC}}{R_1+sL+\frac{1}{sC}}$$

$$y=\left[\frac{\dfrac{-\frac{R_1}{sC}}{R_1+sL+\frac{1}{sC}}}{1+\dfrac{\frac{R_1+sL}{R_2sC}}{R_1+sL+\frac{1}{sC}}}\right]u_1+\left[\frac{1}{1+\dfrac{\frac{R_1+sL}{R_2sC}}{R_1+sL+\frac{1}{sC}}}\right]U_2$$

$$=\left[\frac{-\frac{R_1}{sC}}{R_1+sL+\frac{1}{sC}+\frac{R_1+sL}{R_2sC}}\right]u_1+\left[\frac{R_1+sL+\frac{1}{sC}}{R_1+sL+\frac{1}{sC}+\frac{R_1+sL}{R_2sC}}\right]U_2$$

Example 24

Multiplying top & bottom by s/L yields:

$$y = \left[\frac{-\frac{R_1}{LC}}{\frac{R_1}{L}s + s^2 + \frac{1}{LC} + \frac{R_1+sL}{R_2LC}}\right]u_1 + \left[\frac{\frac{R_1}{L}s + s^2 + \frac{1}{LC}}{\frac{R_1}{L}s + s^2 + \frac{1}{LC} + \frac{R_1+sL}{R_2LC}}\right]U_2$$

$$= \left[\frac{-\frac{R_1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}}\right]u_1 + \left[\frac{s^2 + \frac{R_1}{L}s + \frac{1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}}\right]U_2$$

This is what we got earlier via KVL, KCL, Ohm!

[2] Find transfer functions $H_{1,2}$ from $u_{1,2}$ to y.

$$H_1 = \frac{-\frac{R_1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}}$$

$$H_2 = \frac{s^2 + \frac{R_1}{L}s + \frac{1}{LC}}{s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}}$$

[3] Determine the differential eq relating y to $u_{1,2}$.

$$\ddot{y} + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)\dot{y} + \left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}y = -\frac{R_1}{LC}u_1 + \ddot{u}_2 + \frac{R_1}{L}\dot{u}_2 + \frac{1}{LC}u_2$$

Example 24

4 **Characteristic eq (polynomial):**

$$\Phi(s) = \text{denominator of } H_1 \text{ or } H_2$$

$$= s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)s + \left(1 + \frac{R_1}{R_2}\right)\frac{1}{LC}$$

Note:

Any 2nd order characteristic polynomial may be written as

$$\Phi(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

where $\omega_n \triangleq$ undamped natural frequency

$\xi \triangleq$ damping factor

For our characteristic polynomial

$\omega_n^2 = \left(1 + \frac{R_1}{R_2}\right)\frac{1}{LC} \Rightarrow$ $$\omega_n = \sqrt{1 + \frac{R_1}{R_2}}\left(\frac{1}{\sqrt{LC}}\right)$$

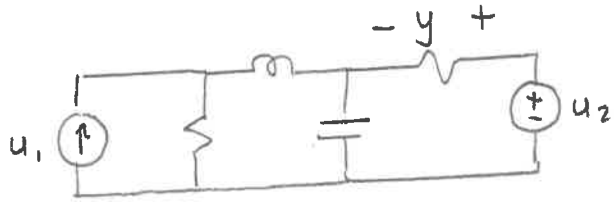
$2\xi\omega_n = \frac{R_1}{L} + \frac{1}{R_2 C} \Rightarrow$ $$\xi = \frac{\frac{R_1}{L} + \frac{1}{R_2 C}}{2\sqrt{1 + \frac{R_1}{R_2}}\left(\frac{1}{\sqrt{LC}}\right)}$$

5 **poles (characteristic roots):**

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Problem 24 (2nd order RLC Cct with 2 Sources) 1290



(All R, L, C = 1)

1) Relate y to u_1, u_2 (in s-domain).

2) Determine transfer functions H_1, H_2 from u_1 to y_1 & from u_2 to y_2

3) Determine diff eq relating y to u_1, u_2.

4) Determine system characteristic equation, poles, stability of system, time constant τ, settling time t_s.

5) Determine y_{ss} when $u_1 = 10 - 3\cos(\sqrt{2}t + 90°) + \sin(0.01t)$

$u_2 = -4 + \sin t - \cos(200t + 45°)$

Example 25

(2nd order systems & damping factor ξ)
(RLC Series Cct)

1300

As stated above,

Any 2nd order characteristic polynomial may be written as:

$$\Phi(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

where

$\omega_n \triangleq$ undamped natural frequency

$\xi \triangleq$ damping factor

The associated characteristic roots (system poles) are:

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

3 Special Cases Are Universally Considered.

... lets examine the 3 special cases ---

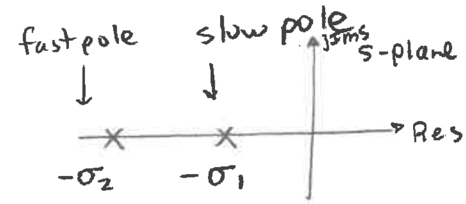
Case 1: $\boxed{\text{Overdamped} \quad (\xi > 1)}$

For this case, the poles are

$$-\sigma_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2-1}, \qquad -\sigma_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

As such, the poles are

real
stable
distinct



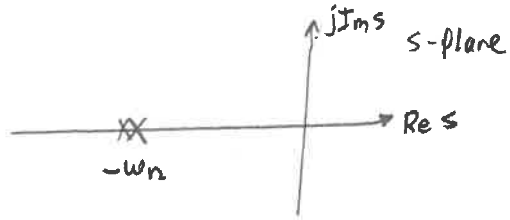
The associated settling time is $t_s = \frac{5}{\sigma_1}$ (associated with the slow pole)

Example 25

Case 2 : Critically Damped ($\zeta = 1$)

For this case, the poles are

$$-\omega_n, -\omega_n$$



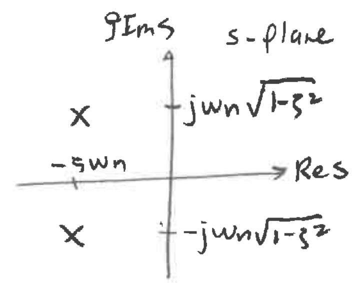
As such, the poles are

real
stable
repeated

The associated settling time is $t_s = \dfrac{5}{\omega_n}$

Case 3 : Under Damped ($0 < \zeta < 1$)

For this case, the poles are



$$s_{1,2} = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$$

$\sigma_o = \zeta\omega_n$
$\omega_o = \omega_n\sqrt{1-\zeta^2}$

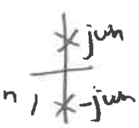
$$= -\sigma_o \pm j\omega_o$$

As such, the poles are

complex
stable
conjugate

This is why ζ is called the undamped natural frequency of the system.

Note: For $\zeta = 0$, the system is undamped, the poles are purely imaginary $s_{1,2} = \pm j\omega_n$, & the system is marginally stable!

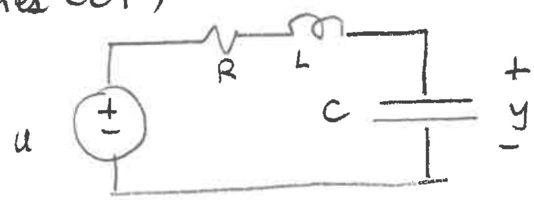


The associated settling time is

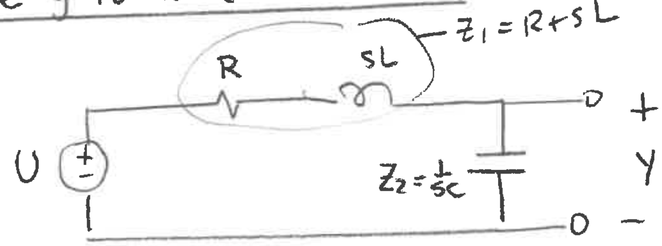
$$t_s = \frac{5}{|\text{Re poles}|} = \frac{5}{|-\sigma_o|} = \frac{5}{\sigma_o} = \frac{5}{\zeta\omega_n}$$

Example 25 (RLC Series Cct)

Consider the RLC cct:



1 Relate y to u (in s-domain)



$$Y = \left(\frac{Z_2}{Z_1 + Z_2}\right) U = \left(\frac{\left(\frac{1}{sC}\right)}{(R+sL) + \left(\frac{1}{sC}\right)}\right) U \;\frac{\frac{s}{L}}{\frac{s}{L}}\!=\; \left[\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right] U$$

2 What is transfer function H from u to y?

$$H = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

3 Determine diff eq relating y to u

$$\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y = \frac{1}{LC}u$$

4 Char eq (poly) =

$$\Phi(s) = \text{denominator of } H$$
$$= s^2 + \frac{R}{L}s + \frac{1}{LC}$$

If we write $\Phi(\cdot)$ as

$$\Phi(s) = s^2 + 2\zeta w_n s + w_n^2$$

determine w_n, ζ.

Note:
$\zeta = 1$ for
$R = R_c = 2\sqrt{\frac{L}{C}}$

$$w_n^2 = \frac{1}{LC} \implies w_n = \frac{1}{\sqrt{LC}}$$

$$2\zeta w_n = \frac{R}{L} \implies \zeta = \frac{R}{2L w_n} = \frac{R}{2L}\sqrt{LC} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Example 25 (RLC Series Cct)

1330

5) poles (characteristic roots):

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

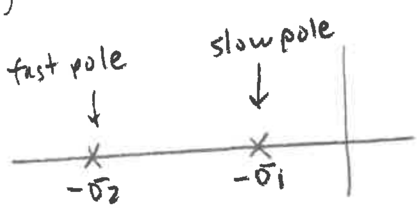
We now consider 3 cases for our circuit:

occurs for ↓ $(R > R_c)$

Case 1: $\boxed{\text{Overdamped } (\zeta > 1)}$

For this case, the poles are

$$-\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\ , \quad -\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$



⟹ The poles are real, stable, & distinct (different).

The associated settling time is

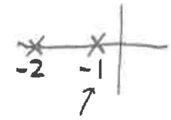
$$t_s = \frac{5}{\sigma_1}$$

ζ is associated with the system's "slow pole".

$R_c = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{1}{\frac{1}{2}}} = 2\sqrt{2} < R = 3$

Here are representative numbers: $R = 3 \quad L = 1 \quad C = \frac{1}{2}$

$$H = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)}$$



$\omega_n = \sqrt{2} \Rightarrow \zeta = \frac{3/2}{\omega_n} = \frac{1.5}{\sqrt{2}} > 1$

$2\zeta\omega_n = 3$

$t_s = 5\,\text{sec}$

Example 25 (RLC Series cct)

Case 2: $\boxed{\text{Critically Damped } (\zeta = 1)}$ occurs for $(R = R_c)$

For this case, the poles are

$$-\omega_n, -\omega_n$$

$\Rightarrow$ The poles are real, stable, repeated (identical).

The associated settling time is

$$t_s = \frac{5}{\omega_n}$$

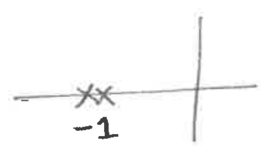
Here are representative numbers: $R = 2 \quad L = C = 1$

$$R_c = 2\sqrt{\frac{L}{C}} = 2 = R$$

$$\Rightarrow \quad H = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{1}{s^2 + 2s + 1} = \left[\frac{1}{s+1}\right]^2$$

$\omega_n = 1$

$2\zeta\omega_n = 2 \Rightarrow \zeta = 1$



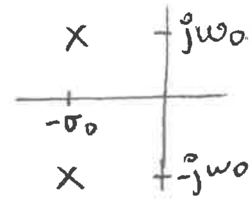
Example 25 (RLC Series Cct)

Case 3 = $\boxed{\text{Under Damped } (0 < \zeta < 1)}$

Occurs for $\downarrow$ $(0 < R < R_c)$

For this case, the poles are

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\sigma_0 \pm j\,\omega_0$$



As such, the poles are complex, stable, & conjugates of one another.

The associated settling time is

$$t_s = \frac{5}{|\text{Re poles}|} = \frac{5}{\sigma_0}$$

Here are representative numbers = $R = L = C = 1$

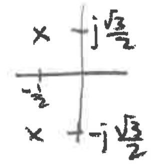
$$R_c = 2\sqrt{\tfrac{L}{C}} = 2 > R = 1$$

$$H = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{1}{s^2 + s + 1}$$

$\omega_n = 1$

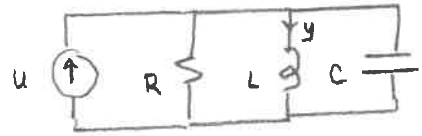
$2\zeta\omega_n = 1 \Rightarrow \zeta = \frac{1}{2} < 1$

$$\text{poles} = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$



$$t_s = \frac{5}{|\text{Re poles}|} = \frac{5}{\frac{1}{2}} = 10 \text{ sec}$$

Problem 25 (RLC Parallel Cct) 1360



1. Relate y to u (in s-domain)
2. Find transfer function H from u to y & diff eq relating y to u
3. Determine system characteristic polynomial, poles, stability.

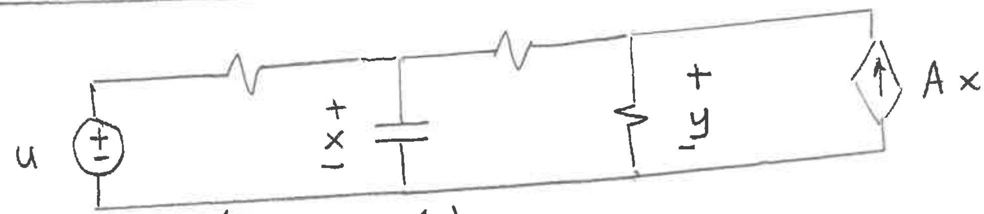
Let $L = C = 1$

4. Determine R such that
   - a) poles $= -\frac{1}{2}, -2$
   - b) poles $= -1, -1$
   - c) poles $= -1 \pm j1$

Let R be that for case c.

5. Determine y_{ss} when $u(t) = A + B\cos(0.01t - 50°) - C\sin(t + 190°) + D\cos(100t + 135°)$

Example 26 (RC Cct with Dependent Source)

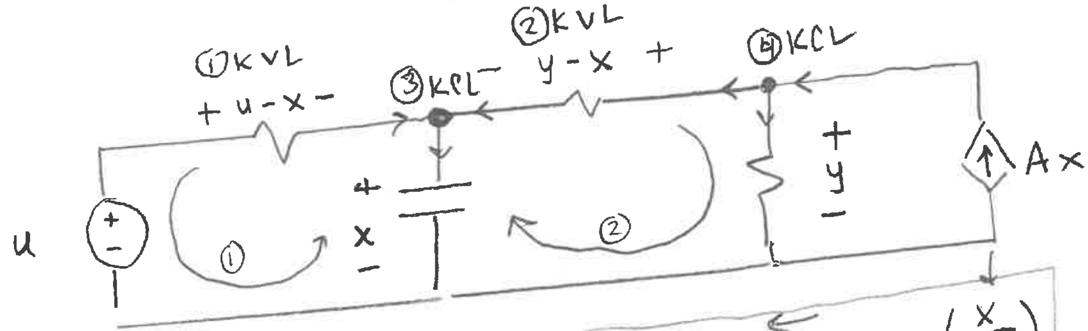


(All R, C = 1)

1 Relate y to u (in s-domain)

Need 2 eqs in 2 unknowns (x, y)

Use one to eliminate x & ~~then~~ solve for y!



③ KCL = (+ Ohm)

$$\boxed{\left(\frac{u-x}{1}\right) + \left(\frac{y-x}{1}\right) = \left(\frac{x}{\frac{1}{s}}\right)}$$

$$u + y = x\,[s+2]$$

④ KCL = (+ Ohm)

$$\boxed{Ax = \left(\frac{y-x}{1}\right) + \left(\frac{y}{1}\right)}$$

$$\Rightarrow (A+1)x = 2y$$

$$\Rightarrow x = \left(\frac{2}{A+1}\right) y$$

$$\Rightarrow u + y = \left(\left(\frac{2}{A+1}\right) y\right)[s+2]$$

$$\Rightarrow \left(\frac{A+1}{2}\right) u + \left(\frac{A+1}{2}\right) y = y\,[s+2]$$

$$\Rightarrow y\left[s+2-\frac{A+1}{2}\right] = \left(\frac{A+1}{2}\right) u$$

$$\Rightarrow y\,[2s+4-A-1] = (A+1)\,u$$

$$y\,[2s+3-A] = (A+1)\,u$$

$$\Rightarrow \quad y = \left[\frac{\left(\frac{A+1}{2}\right)}{s + \left(\frac{3-A}{2}\right)} \right] u$$

[2] Determine transfer function H from u to y.

$$H(s) = \frac{\frac{A+1}{2}}{s + \left(\frac{3-A}{2}\right)}$$

[3] Determine diff eq relating y to u.

$$\dot{y} + \left(\frac{3-A}{2}\right) y = \left(\frac{A+1}{2}\right) u$$

[4] Determine system characteristic eq & pole.

$$\Phi(s) = s + \left(\frac{3-A}{2}\right) = 0$$

$$\Rightarrow \quad \text{pole} = -\left[\frac{3-A}{2}\right]$$

Note: System is stable $\overset{\text{iff}}{\Longleftrightarrow}$ $A < 3$ (so that pole < 0)

marginally stable $\overset{\text{iff}}{\Longleftrightarrow}$ $A = 3$ (so that pole $= 0$)

unstable $\overset{\text{iff}}{\Longleftrightarrow}$ $A > 3$ (so that pole > 0)

Lets suppose that $A < 3$ so that our system pole $= -\left(\frac{3-A}{2}\right) < 0$ & hence the system is stable!

$\boxed{\text{Example 26}}$ (RC Cct with Dependent Source)

$\boxed{5}$ Determine the system time constant τ & settling time t_s.

$$\tau = \frac{1}{|\,Re\,(\text{slowest pole})\,|} = \frac{1}{\left|\,Re\left(-\left(\frac{3-A}{2}\right)\right)\right|}$$

$$= \frac{1}{\left|-\left(\frac{3-A}{2}\right)\right|}$$

$$= \frac{1}{\frac{3-A}{2}}$$

$$= \frac{2}{3-A} \quad (> 0 \text{ since } A < 3)$$

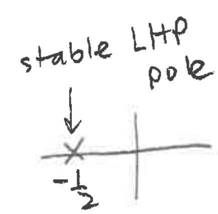
$$t_s = 5\tau = \frac{5}{|\,Re\,(\text{slowest pole})\,|} = \frac{10}{3-A}$$

Note: One could choose A in order to achieve a specific t_s!

... that is design!!!

Now let $A = 2$.

This yields
$$H(s) = \frac{\frac{A+1}{2}}{s + \left(\frac{3-A}{2}\right)} = \frac{\frac{2+1}{2}}{s + \left(\frac{3-2}{2}\right)} = \frac{\frac{3}{2}}{s + \frac{1}{2}}$$



$\Rightarrow \tau = 2$

$t_s = 5\tau = 10\text{ sec}$

Example 26 (RC Cct with Dependent Source)

[6] Determine y_{ss} when $u = A + B\sin(0.05t + 25°) + C\cos(\frac{1}{2}t + 135°) + D\sin(50t + 120°)$

By MOTF:

$$y_{ss} = A\,H(0) + B\,|H(j0.05)|\sin(0.05t + 25° + \underline{/H(j0.05)})$$
$$+ C\,|H(j\tfrac{1}{2})|\cos(\tfrac{1}{2}t + 135° + \underline{/H(j\tfrac{1}{2})})$$
$$+ D\,|H(j50)|\sin(50t + 120° + \underline{/H(j50)})$$

Recalling that

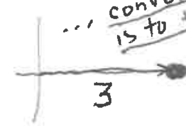
$$H(s) = \frac{3/2}{s + 1/2}$$

we now compute each of the quantities (involving H), required above.

$$H(0) = \frac{3/2}{1/2} = 3$$

Magnitude = 3
Angle = 0 (± integer multiple of 360°)
... convention is to take 0° as angle of any positive real #!

$$H(j0.05) = \frac{3/2}{j0.05 + \underbrace{(1/2)}_{0.5}} \cong \frac{3/2}{0 + \underbrace{(1/2)}_{0.5}} = 3 = \underbrace{3}_{|H(j0.05)|}e^{j\overbrace{0}^{/H(j0.05)}}$$



$\frac{1}{2} = 0.5$

j0.05 – 10 times smaller than 0.5 (hence neglible)

$$H(j50) = \frac{3/2}{j50 + (\frac{1}{2})} \cong \frac{3/2}{j50} = \frac{3/2}{50e^{j90°}} = \underbrace{0.03}_{|H(j50)|}e^{\overbrace{-j90°}^{/H(j50)}}$$

polar form

$0.5 = \frac{1}{2}$

50 = 100 times larger than $\frac{1}{2} = 0.5$

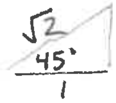
$j50 = 50e^{j90°}$

rectangular form

$\boxed{\text{Example 26}}$ $\boxed{1410}$

$$|H(j\tfrac{1}{2})| \qquad \angle H(j\tfrac{1}{2})$$

$$H(j\tfrac{1}{2}) = \frac{3/2}{j\frac{1}{2}+\frac{1}{2}} = \frac{\cancel{\frac{1}{2}}\,(3)}{\cancel{\frac{1}{2}}\,(j1+1)} = \frac{3}{1+j1} = \frac{3}{\sqrt{2}e^{j45°}} = \frac{3}{\sqrt{2}}e^{-j45}$$



$$1+j1 = \sqrt{2}\,e^{j45°}$$

rectangular form — polar form

On an exam, the above suffices!

Here, for completeness, I shall show where everything we computed goes:

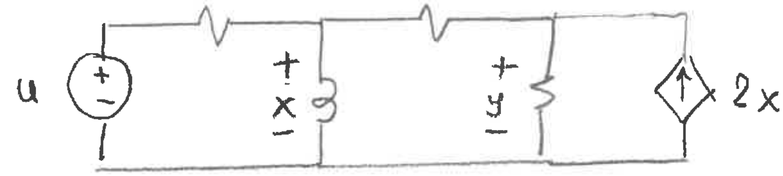
$$y_{ss} = A\,\underbrace{H(0)}_{3} + B\,\underbrace{|H(j0.05)|}_{\sim 3}\sin(0.05t + 25 + \underbrace{\angle H(j0.05)}_{\sim 0°})$$

$$+ C\,\underbrace{|H(j\tfrac{1}{2})|}_{\frac{3}{\sqrt{2}}}\cos(\tfrac{1}{2}t + 135 + \underbrace{\angle H(j\tfrac{1}{2})}_{-45°})$$

$$+ D\,\underbrace{|H(j50)|}_{\sim 0.03}\sin(50t + 120° + \underbrace{\angle H(j50)}_{\sim -90°})$$

Please review your complex arithmetic!

Problem 26 (RL Cct with Dependent Source) 1420

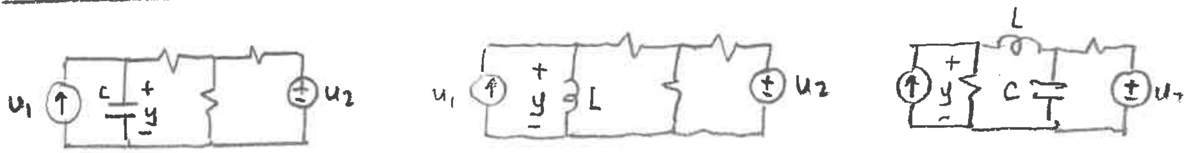


(All R = L = 1)

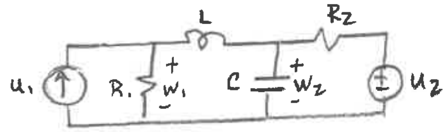
1. Relate y to u (in s-domain)
2. Determine transfer function H from u to y & diff eq
3. Determine system characteristic polynomial, pole, stability, time constant τ, & settling time t_s
4. Determine y_{ss} when $u = A - B\cos(0.02t + 30^\circ) + C\sin(2t + 45^\circ) - D\cos(300t + 90^\circ)$

Circuits You Should Examine

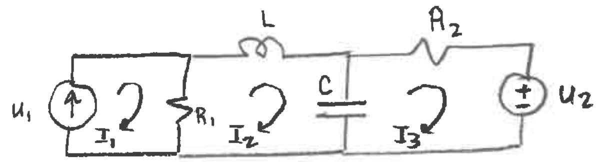
$\boxed{10}$ pg 270



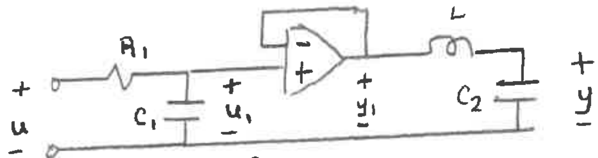
$\boxed{11}$ Nodal pg 310



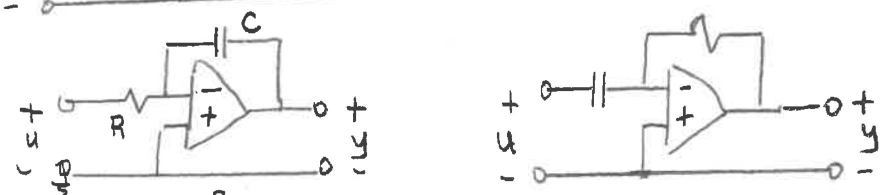
$\boxed{12}$ Mesh pg 340



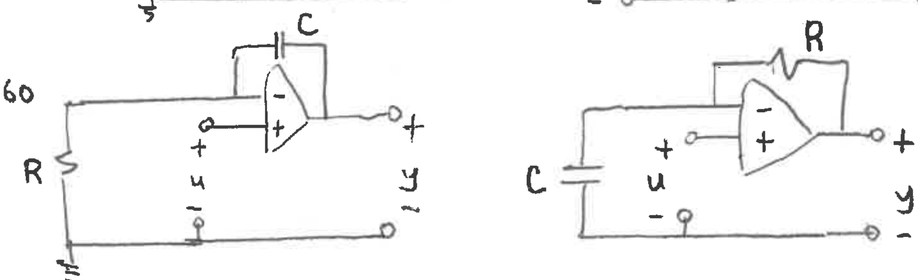
$\boxed{13}$ pg 410



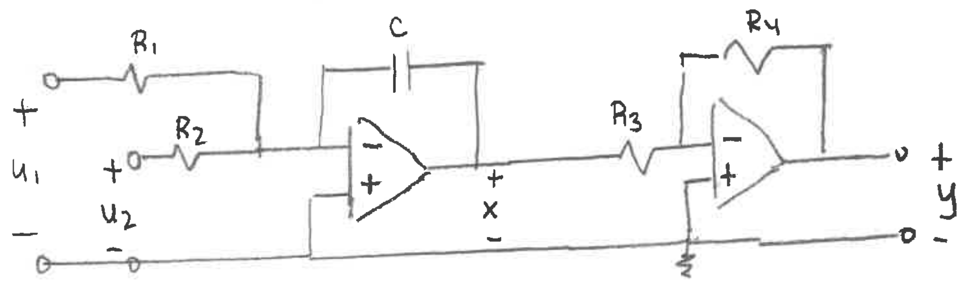
$\boxed{14}$ pg430



$\boxed{15}$ pg 460

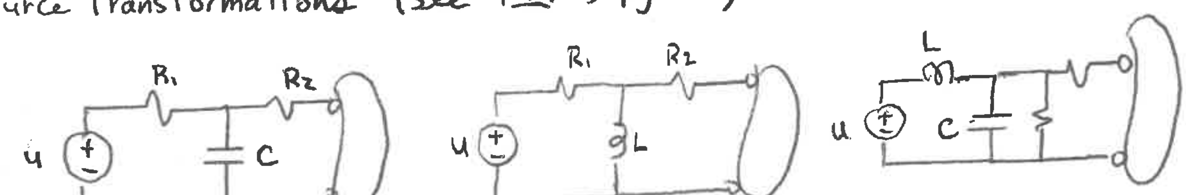


$\boxed{16}$ pg480

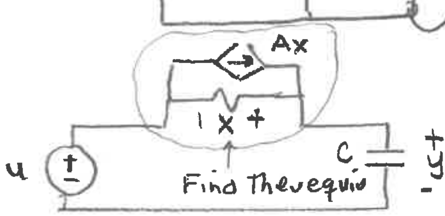


$\boxed{17}$ pg 510 Source Transformations (see $\boxed{7}$ & pg 170)

$\boxed{18}$ Thevenin pg 540



$\boxed{19}$ pg 610

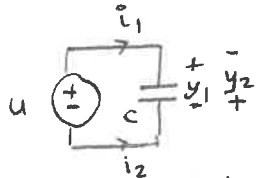


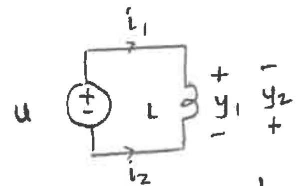
$A = \frac{1}{R} - \frac{1}{Req}$

Circuits You Should Examine

Like Example 1

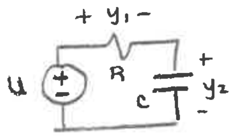
1 pg 20

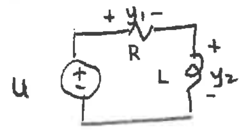


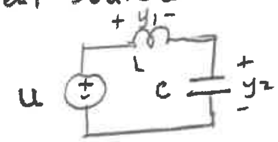


examine with current sources too!

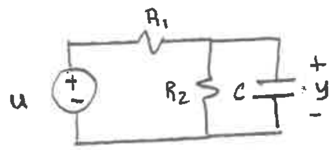
2 pg 50

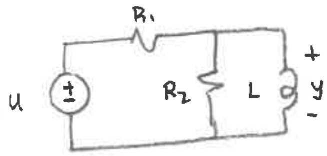


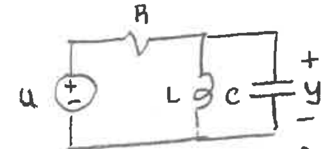




3 pg 70







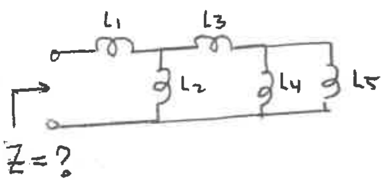
... can move L & C around!

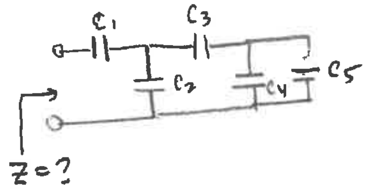
like Problem 3

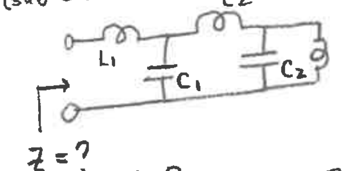
P3 pg 100

(sub C with L too!)

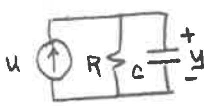
4 pg 110

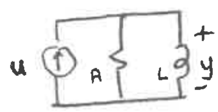


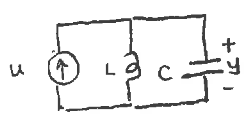




5 pg 130

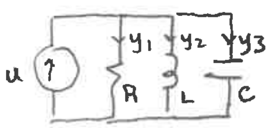


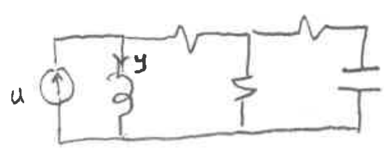




P5 pg 140

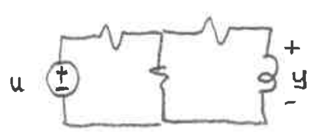
6 pg 150

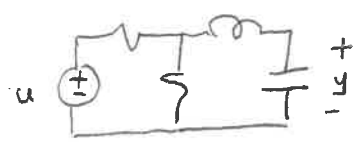




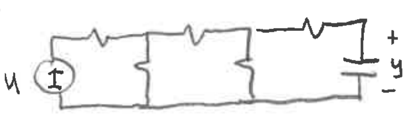
pg 160

7 pg 170



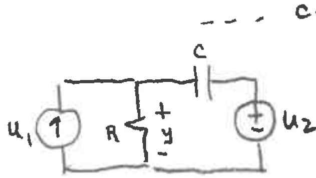


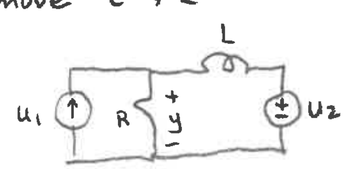
8 pg 220

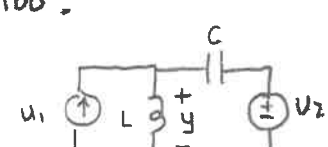


... can move C & L around too!

9 pg 240



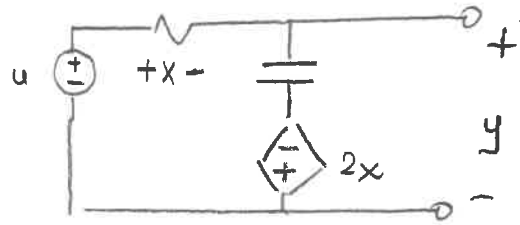




... can move C & L around too!

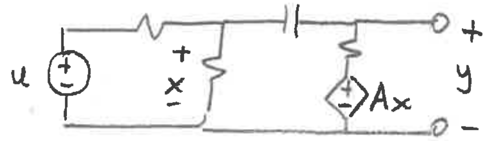
Circuits You Should Examine

$\boxed{20}$ pg 720

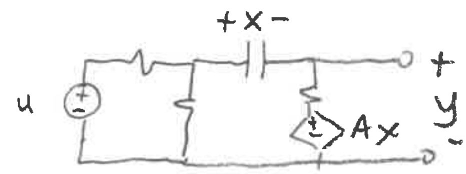


R = C = 1

$\boxed{P20}$ pg 820



$R_i = C = 1$ (relate y to u)



$R_i = C = 1$ (relate y to u)

$\boxed{21}$ pg 830