```
xinit = 1.165561;                  % initial condition
xinc = 0.000000001;                % initial condition increment
xfin = 1.1655611859;               % final initial condition
ic = [xinit:xinc:xfin];            % vector of inital conditions
roots = 0*[xinit:xinc:xfin];       % space for roots
m = 0;                             % initialize initial condition counter
numit = 10000;                     % number of Newton iterations to be performed
for x = xinit:xinc:xfin,
    m = m + 1;                     % increment initial condition counter
    for k = 1:1:numit              % perform numit Newton iterations
        x = x - tan(x);            % perform Newton iteration
    end
    roots(1,m) = x;                % update root
end
plot(ic, roots/pi);                % plot roots obtained versus ic used
grid.
return
```

The following summarizes our investigations.

Specific Initial Condition. *If* $x_o = \pi/6 = 0.5236$, *then this relation results in the following sequence:* $x_1 = -0.0538$, $x_2 = 5.1827 \times 10^{-5}$, $x_3 = -4.6402 \times 10^{-14}$, ... *This confirms that* $x = 0$ *is a root of* $\sin x = 0$. *It also shows that Newton's algorithm can converge quite rapidly when the initial condition* x_o *is appropriately selected.*

Other Initial Conditions. *For* $x_o \in [0, 1.1655611859]$, *the above recursion converges toward the root at* $x = 0$. *For larger values of* x_o, *problems start occurring. For* $x_o = 1.165561186 \approx 0.371\pi$, *for example, the recursion converges to the root at* $x = -32\pi$. ∎

C.3 Elements from Complex Variables

Complex numbers are particularly important in the study of linear dynamical systems. Their main utility lies in the fact that they permit engineers to *represent sinusoidal and exponential sinusoidal functions by complex exponentials*. Given this, complex exponentials greatly facilitate the study of linear systems. Moreover, their use leads naturally to new powerful system concepts (e.g. frequency response of a system, etc.). As such, complex numbers are indispensable in the study of dynamical systems.

Motivation: The Need For Complex Numbers. Because equations of the form

$$z^2 = -1 \tag{C.61}$$

were found to be present everywhere in science and engineering, it became essential to invent *complex numbers*. (Necessity is the mother of invention!)

Definition C.3.1 (Complex Numbers)

A complex number z *is an ordered pair of real numbers* x *and* y. *The complex number* z *is denoted* $z \stackrel{\text{def}}{=} (x, y)$. *The real number* x *is called the* real part *of the complex number* $z = (x, y)$. *The real number* y *is called the* imaginary part *of the complex number* $z = (x, y)$. *If* $x = 0$, *we say that the complex number* $z = (0, y)$ *is* purely imaginary. *If* $y = 0$, *we say that the complex number* $z = (x, 0)$ *is* real. *The set of all complex numbers is denoted* C.

Equality of Complex Numbers. *Two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *are equal if and only if their real parts are equal and their imaginary parts are equal; i.e.*

$$z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2. \tag{C.62}$$

Addition of Complex Numbers. *The sum of two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *is denoted* $z_1 + z_2$ *and defined as follows:*

$$z_1 + z_2 \stackrel{\text{def}}{=} (\ x_1 + x_2,\ y_1 + y_2\ ). \tag{C.63}$$

That is, the sum is obtained by adding the real parts and by adding the imaginary parts to get the new real and imaginary parts, respectively. This definition implies that complex numbers add like vectors. An intuitive geometric interpretation is thus possible.

Multiplication of Complex Numbers. *The product of two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *is denoted* $z_1 z_2$ *and defined as follows:*

$$z_1 z_2 \stackrel{\text{def}}{=} (\ x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1\ ). \tag{C.64}$$

While this definition is easy to use, it is rather strange and arguably unmotivated. An intuitive (geometric) interpretation for complex multiplication will be given below. ∎

Commutative, Associative, and Distributive Properties. It can be shown that the addition and multiplication of complex numbers are *commutative*, *associative*, and *distributive* operations. That is, for any complex numbers z_1, z_2, and z_3, the following relationships hold:

$$\begin{array}{lcllcl} z_1 + z_2 & = & z_2 + z_1 & z_1 z_2 & = & z_2 z_1 \\ (z_1 + z_2) + z_3 & = & z_1 + (z_2 + z_3) & (z_1 z_2) z_3 & = & z_1 (z_2 z_3) \\ z_1 (z_2 + z_3) & = & z_1 z_2 + z_1 z_3 & (z_1 + z_2) z_3 & = & z_1 z_3 + z_2 z_3. \end{array} \tag{C.65}$$

Most of us take these properties for granted when working with real numbers. It is very comforting that they hold for complex numbers as well.

The Symbol j. Given the above definition for complex multiplication, it follows that $(0,1)(0,1) = (-1,0) = -1$ and $(0,-1)(0,-1) = (0,1)(0,1) = (-1,0) = -1$. From this, it follows that $z = (0,1)$ and $z = (0,-1) = -(0,1)$ are each solutions to the quadratic equation $z^2 = -1$. Because of this, we assign the special symbol j to $(0,1)$; i.e.

$$j \stackrel{\text{def}}{=} (0,1). \tag{C.66}$$

Because $j^2 = -1$, it follows that $j = \sqrt{-1}$. It should be noted that while the symbol j is widely used within the engineering community, the symbol i is still the preferred symbol within the mathematics community [1], [8], [261]. The symbol i is used to represent electrical current within the engineering community [139].

Rectangular Representation. With the above "j-notation" in place, it follows that

$$(x,y) = (x,0) + (0,y) = (x,0) + (0,1)(y,0) = (x,0) + j(y,0) = x + jy \tag{C.67}$$

This provides motivation for the widely used notation

$$z = x + jy \tag{C.68}$$

which we adopt hereafter. This representation for z is commonly referred to as the *rectangular representation* for z. Given this, the complex number $z = x + jy$ can be visualized as indicated in Figure C.1. The idea of representing complex numbers by points (vectors) in the plane was formulated by Gauss in his 1799 dissertation and, independently, by Argand in 1806 [8, pg. 17]. Figure C.1 is sometimes referred to as an

Argand diagram.

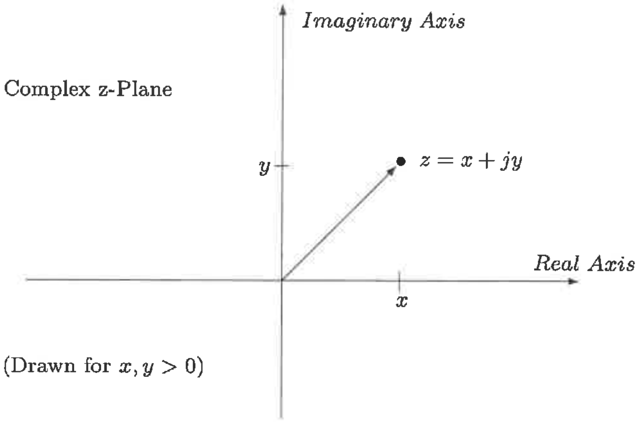

Figure C.1: Visualizing a Complex Number z In The Complex z-Plane

With the above notation, equations (C.63) and (C.64) become

$$z_1 + z_2 \stackrel{\text{def}}{=} [x_1 + x_2] + j\,[y_1 + y_2] \tag{C.69}$$

$$z_1 z_2 \stackrel{\text{def}}{=} [x_1 x_2 - y_1 y_2] + j\,[x_1 y_2 + x_2 y_1]\,. \tag{C.70}$$

In order to properly define complex division, it is useful to introduce new terms.

Conjugate of a Complex Number. The *conjugate* of a complex number $z = x + jy$ is denoted $\overline{z}$ and defined as follows:

$$\overline{z} \stackrel{\text{def}}{=} x - jy. \tag{C.71}$$

The conjugate $\overline{z}$ of the complex number $z = x + jy$ may be visualized as shown if Figure C.2. Moreover, it can be shown that

$$\overline{\overline{z}} = z. \tag{C.72}$$

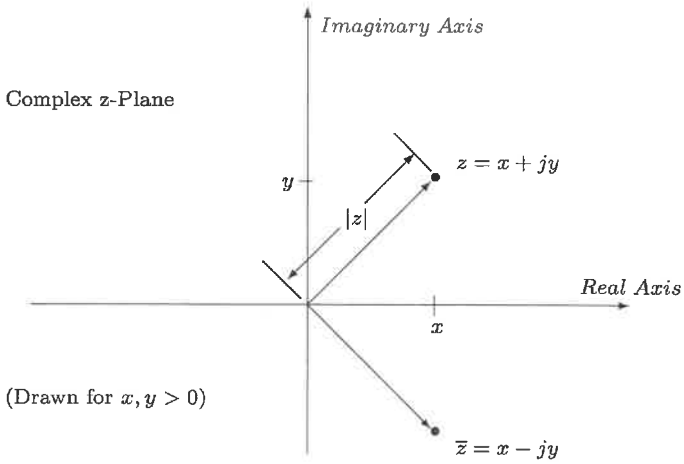

Figure C.2: Visualizing The Conjugate $\overline{z}$ Of A Complex Number z In The Complex z-Plane

Magnitude of a Complex Number. The *magnitude* of a complex number $z = x + jy$ is denoted $|z|$ and defined as follows:

$$|z| \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}. \tag{C.73}$$

The magnitude $|z|$ can be visualized as shown in Figure C.2. From the figure, and the Pythagorean theorem, it follows that $|z|$ is precisely the length of the vector used in representing z. It is useful to note that

$$|z| = \sqrt{z\,\overline{z}} \tag{C.74}$$

Division. Given the above, the division of two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ ($z_2 \neq 0$) is defined in terms of the ratio

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \tag{C.75}$$

as follows

$$\frac{z_1}{z_2} \stackrel{\text{def}}{=} \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right] + j\left[\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right]. \tag{C.76}$$

As with complex multiplication, an intuitive (geometric) interpretation for complex division will be given below.

The rectangular representation for complex numbers is most useful for carrying out and visualizing the addition and subtraction of complex numbers. Rectangular representations, however, are not very intuitive when it comes to carrying out and visualizing complex multiplication and division. To obtain nicer formulae - with useful geometric interpretation, we need to motivate an alternative representation for complex numbers. Recalling that real exponentials satisfy very nice relationships for multiplication and division, we are led naturally to *polar representations* and *complex exponentials*. To properly introduce these ideas, we require a new term - the *angle* of a complex number.

Angle of a Complex Number. Consider the complex number $z = x + jy$ shown in Figure C.3. The *angle* of the complex number z is defined as follows:

$$\angle z \stackrel{\text{def}}{=} \begin{cases} \tan^{-1}\frac{y}{x} & x > 0,\quad y > 0 \\ 180 - \tan^{-1}\frac{y}{|x|} & x < 0,\quad y > 0 \\ -180 + \tan^{-1}\frac{|y|}{|x|} & x < 0,\quad y < 0 \\ -\tan^{-1}\frac{|y|}{x} & x > 0,\quad y < 0 \end{cases} \tag{C.77}$$

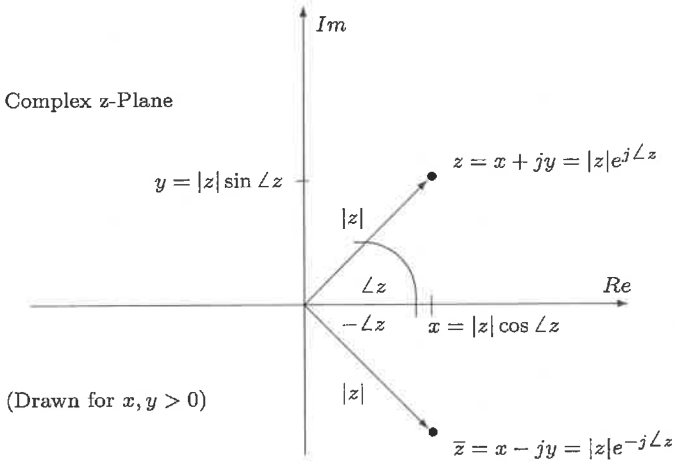

Figure C.3: Visualizing the Polar Form of a Complex Number $z = x + jy$

Comment C.3.1 (Issues with Angle of a Zero)
We note that the angle of a complex number z only makes sense if $z \neq 0$. If $|z| = 0$, we say that the angle is indeterminate *or* undefined.

Zero Magnitude Resulting from Limiting Process. *If $|z| = 0$ is obtained as the result of some limiting process, however, then care must be taken. To illustrate this, consider a positive scalar $c > 0$ and the complex number $z = c(1 + j1) = c\sqrt{2}e^{j45^\circ}$. Given this, we see that $\lim_{c \to 0^+} \angle z = 45^\circ$. That is, the angle in the limit is well defined even though the magnitude approaches zero in the limit. Hence the need for care!* ■

It should be noted that engineers often use the word *phase* instead of angle.

From Figure C.3, it follows that the real and imaginary parts of a complex number z may be readily computed from the magnitude and angle of z as follows:

$$Re\{z\} = |z| \cos \angle z \qquad\qquad Im\{z\} = |z| \sin \angle z. \tag{C.78}$$

This motivates the alternative representation that we seek for complex numbers - the so-called *polar* or *exponential representation.*

Polar (Exponential) Representation. From the above discussion, it follows that

$$z = |z| e^{j\angle z} \tag{C.79}$$

where

$$e^{j\angle z} \stackrel{\text{def}}{=} \cos \angle z + j \sin \angle z. \tag{C.80}$$

Equation (C.79) defines the so-called *polar* or *exponential representation* for the complex number z.

Exercise C.3.1 (Scaling a Complex Number by a Real Scalar)
Let c denote any real scalar and z any complex number. Show that

$$\begin{aligned} Re\{\, c\, z \,\} &= c\, Re\{\, z \,\} && \text{(C.81)} \\ Im\{\, c\, z \,\} &= c\, Im\{\, z \,\} && \text{(C.82)} \\ |\, c\, z \,| &= |\, c \,|\,|\, z \,| && \text{(C.83)} \\ \angle(cz) &= \angle c + \angle z && \text{(C.84)} \end{aligned}$$

where, in the last expression, $\angle c = 0^\circ$ if $c > 0$ and $\angle c = 180^\circ$ if $c < 0$. If $c = 0$, then $\angle c$ is undefined and the last relationship is meaningless. In such a case, the prior three relationships still hold. ■

Polar Representation Properties: Geometric Interpretations For Multiplication And Division. The polar representation of a complex number offers very nice properties when it comes to complex multiplication and division. If $z_1 = |z_1| e^{\angle z_1}$ and $z_2 = |z_2| e^{\angle z_2}$, then it can be shown that

$$\begin{aligned} z_1 z_2 &= |z_1||z_2|\, e^{j(\angle z_1 + \angle z_2)} && \text{(C.85)} \\ \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|}\, e^{j(\angle z_1 - \angle z_2)}. && \text{(C.86)} \end{aligned}$$

Unlike their rectangular counterparts (cf. equations (C.70), (C.76)), these properties are very useful and are readily visualized geometrically. Equation (C.85) shows that the multiplication of two complex numbers involves the multiplication of their magnitudes and the addition of their angles. Equation (C.86) shows that the division of two complex numbers involves the division of their magnitudes and the subtraction of their angles.

Euler Identities. From the above discussion and Figure C.4, one obtains the following Euler identities:

$$z + \bar{z} = 2Re\{z\} = 2|z| \cos \angle z \tag{C.87}$$
$$z - \bar{z} = j2Im\{z\} = 2|z| \sin \angle z. \tag{C.88}$$

These identities are extensively used in science and engineering.

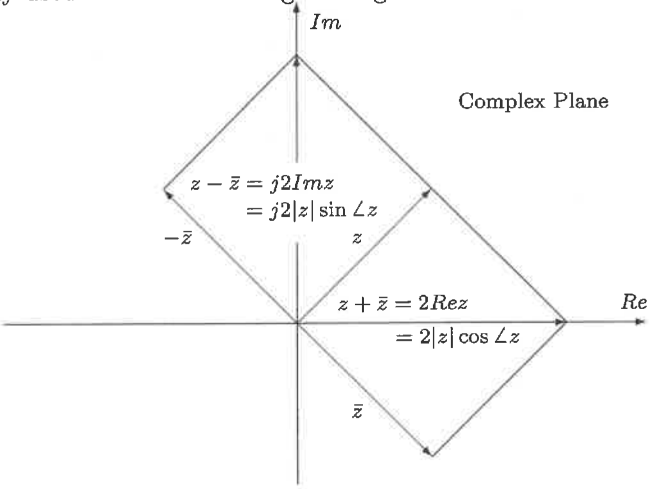

Figure C.4: Visualizing Euler's Identities

Historical Note. The Swiss mathematician Leonhard Euler (1707-1783) was taught by Johann Bernoulli - father of Daniel Bernoulli (1700-1782). Euler entered the university at age 13 and received his masters degree at age 16. He collaborated with Daniel on problems in fluid mechanics - deriving the so-called *Bernoulli equation* which relates the speed of a fluid to its pressure. He would lose sight in his right eye in 1735 and in his left eye in 1766. Euler is generally regarded as the preeminent mathematician of the 18th century - Lagrange perhaps coming in second.

Complex Exponential, Periodicity. Just as the polar representation of a complex number was motivated by the nice multiplication and division properties that exponentials offer, these ideas motivate the definition for the *complex exponential* e^z of a complex number $z = x + jy$:

$$e^z \stackrel{\text{def}}{=} e^x e^{jy} \tag{C.89}$$

From equation (C.80), it then follows that

$$e^z = e^x e^{jy} = e^x \cos y + je^x \sin y. \tag{C.90}$$

From this, it follows that the magnitude of e^z is given by

$$|e^z| = e^x \tag{C.91}$$

and the angle e^z is given by

$$\angle e^z = y. \tag{C.92}$$

It is very important to note that the complex exponential is periodic in the angle y with period 2π. Thus, e^0 and $e^{j2\pi}$ are different representations for the same number, namely $z = 1$.

Roots. The polar representation of a complex number z is also very useful for determining the roots of the number as follows:

$$z^{\frac{m}{n}} = |z|^{\frac{m}{n}} e^{j\frac{m}{n}\angle z} = |z|^{\frac{m}{n}} e^{j\frac{m}{n}(\angle z + 2\pi k)} \quad k = 0, 1, \ldots n - 1. \tag{C.93}$$

Exercise C.3.2 (Representation of Complex Numbers)
Complex arithmetic comes up in many science and engineering problems (e.g. circuit analysis, system analysis, etc.). The purpose of this exercise is to become complex arithmetic experts. More specifically, we want to learn when we can make approximations. Toward this end, I suggest that you learn to appropriately apply the following "one-plus-ten rules" *as often as possible:*

$$1+10 \approx 10 \qquad \left(\text{if an error of } \frac{11-10}{11} \times 100 \approx 9.1\% \text{ is acceptable}\right) \tag{C.94}$$

$$1+20 \approx 20 \qquad \left(\text{if an error of } \frac{21-20}{21} \times 100 \approx 4.76\% \text{ is acceptable}\right) \tag{C.95}$$

$$1+100 \approx 100 \qquad \left(\text{if an error of } \frac{101-100}{101} \times 100 \approx 1\% \text{ is acceptable}\right). \tag{C.96}$$

One might refer to the above rules as "approximate 10%, 5%, and 1% approximation rules." *While we will typically use the "one-plus-ten" (approximate 10%) rule, we emphasize that how one approximates depends on the application at hand and the associated error tolerance. As an engineer, you would like to get used to making approximations for two reasons: (1) Approximations, by definition, are easier to arrive at than accurate answers, (2) Even coarse approximations can sometimes provide significant insight! Calculators/-computers should be used to check how good your approximations are; do not become too dependent on them! They should also be used when very accurate answers are required.*

For each of the following complex numbers, determine the magnitude, angle, real part, imaginary part, and polar representation. For each case, draw pictures when appropriate. Approximate whenever you can!

(a) $z = 3 + j4$ *Answer:* $|z| = 5$, $\angle z = 53°$, $Re\ z = 3$, $Im\ z = 4$, $z = |z|e^{j\angle z} = 5e^{j53°}$.

(b) $z = -1+j1$ *Answer:* $|z| = \sqrt{2}$, $\angle z = 180° - \tan^{-1}\left(\frac{1}{|-1|}\right) = 180° - 45° = 135°$, $Re\ z = -1$, $Im\ z = 1$, $z = |z|e^{j\angle z} = \sqrt{2}e^{j135°}$. *Strong Advice: Sketch a picture of z to visualize and understand the above angle formula. General Observation: Those who draw, get it right. Those who don't draw, get it wrong. Pictures are particularly important when you are getting started!*

(c) $z = -1 + j\sqrt{3}$ *Answer:* $|z| = 2$, $\angle z = 180° - \tan^{-1}\left(\frac{\sqrt{3}}{|-1|}\right) = 180° - 60° = 120°$, $Re\ z = -1$, $Im\ z = \sqrt{3}$, $z = |z|e^{j\angle z} = 2e^{j120°}$. *Strong Advice: Sketch a picture of z to visualize and understand the above angle formula. General Observation: Those who draw, get it right. Those who don't draw, get it wrong.*

(d) $z = \sqrt{3} - j1$ *Answer:* $|z| = 2$, $\angle z = -30°$, $Re\ z = 2$, $Im\ z = -1$, $z = |z|e^{j\angle z} = 2e^{-j30°}$.

(e) $z = \frac{(\sqrt{3}-j1)(-4-j3)}{(-1+j1)(-2\sqrt{3}-j2)}$

Answer: $z = \frac{(2e^{-j30°})(5e^{-j143°})}{(\sqrt{2}e^{j135°})(4e^{-j150°})}$, $|z| = \frac{(2)(5)}{(\sqrt{2})(4)} = 1.7678$, $\angle z = -30° - 143° - 135° + 150° = -158°$, $Re\ z = |z|\cos\angle z = (1.7678)(-0.9272) = -1.6390$, $Im\ z = |z|\sin\angle z = (1.7678)(-0.3746) = -0.6622$. *Note how each component of z was written in polar form first. A picture for each component (i.e. each complex number in parentheses) will help you get each of the component angles correct. Draw those pictures!*

Perspective: Preparing to Work with System Transfer Functions. *In the study of linear dynamical systems, one encounters the concept of a system transfer function. Continuous-time system transfer functions are written as a function of s - the variable used in the so-called Laplace Transform. When such systems (referred to as linear time invariant or LTI systems) are driven by a sinusoidal forcing function of frequency* ω_o, *it is possible to determine the steady state output (i.e. ouput after a long time) readily using the so-called* Method of the Transfer Function (MOTF). *See Exercise 2.3.1 for an introduction. Also see, Equation (3.144) in Section 3.5. Doing so, however, requires the evaluation of the system transfer function at a point* $s = j\omega$. *The following cases are designed to prepare for this activity.*

(f) $z = \frac{1}{s+1}|_{s=j1}$

(g) $z = \frac{1}{s+1}|_{s=j10}$ *Suggestion: Approximate!*
Partial answer: $z \approx 0.1e^{-j90^\circ}$. *A more accurate result is* $z \approx 0.095e^{-j84.2894^\circ}$.
Approximate magnitude is off by 0.05 *or about* 5.3%. *Approximate angle is off by* 5.71° *or about* 6.8%. *I like the approximate answer better. Don't you? Approximate answers are more readily arrived at and can provide significant insight! Note: It is useful to remember the above* 5.71° *correction.*

(h) $z = \frac{1}{s+1}|_{s=j0.1}$ *Suggestion: Approximate!*
Partial Answer: $z \approx 1$. *A more accurate result is* $z = 0.9950e^{-j5.7106^\circ}$.

(i) $z = \frac{1}{s-1}|_{s=j1}$

(j) $z = \frac{1}{s-1}|_{s=j10}$ *Suggestion: Approximate!*
Partial Answer: $z \approx 0.1e^{-j90^\circ}$. *Comments similar to those made in (g) can be made in this case.*

(k) $z = \frac{1}{s-1}|_{s=j0.1}$ *Suggestion: Approximate!*
Partial Answer: $z \approx \frac{1}{-1} = \frac{1}{1e^{j180^\circ}} = 1e^{-j180^\circ}$

(l) $z = \frac{1}{s(s+1)(s-1)}|_{s=j1}$

(m) $z = \frac{1}{s(s+1)(s-1)}|_{s=j10}$ *Suggestion: Approximate!*
Partial Answer: $z \approx \frac{1}{(j10)(j10)(j10)} = \frac{1}{1000e^{j270^\circ}} = 0.001e^{-j270^\circ}$

(n) $z = \frac{1}{s(s+1)(s-1)}|_{s=j0.1}$ *Suggestion: Approximate!*
Partial Answer: $z \approx \frac{1}{(j0.1)(1)(-1)} = \frac{1}{(0.1e^{j90^\circ})(1)(1e^{j180^\circ})} = 10e^{-j270^\circ}$

(o) $z = \frac{1}{s(s^2+s+1)}|_{s=j1}$

(p) $z = \frac{1}{s(s^2+s+1)}|_{s=j10}$ *Suggestion: Approximate!*

(q) $z = \frac{1}{s(s^2+s+1)}|_{s=j0.1}$ *Suggestion: Approximate!*

(r) $z = \frac{s+1}{s(s^2+s+1)}\ e^{-\pi s}|_{s=j1}$

(s) $z = \frac{s+1}{s(s^2+s+1)}\ e^{-\pi s}|_{s=j10}$ *Suggestion: Approximate!*

(t) $z = \frac{s+1}{s(s^2+s+1)}\ e^{-\pi s}|_{s=j0.1}$ *Suggestion: Approximate!*

(u) $z = \frac{(s+100)^2(s-200)}{s(s+1)^2(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})^3(s+50)^2} e^{-(s+\frac{1}{2})\frac{\sqrt{3}\pi}{2}}|_{s=-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$ *Suggestion: Approximate when you can!*

(v) $z = \frac{(s+100)^2(s-200)}{s(s+1)^2(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})^3(s+50)^2}|_{s=j1000}$ *Suggestion: Approximate when you can!*
$z \approx \frac{1}{s^5}|_{s=1000e^{j90^\circ}} = 10^{-15}e^{-j450^\circ}$.

(w) $z = \frac{(s+100)^2(s-200)}{s(s+1)^2(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})^3(s+50)^2}|_{s=j0.1}$ *Hint: Approximate when you can!*
$z \approx \frac{(100)^2(-200)}{s(1)^2(\frac{1}{2}+j\frac{\sqrt{3}}{2})^3(50)^2}|_{s=j0.1} = \frac{(100)^2(200e^{j180^\circ})}{0.1e^{j90^\circ}(1)^2(1e^{j60^\circ})^3(50)^2}$.

(x) $z = \frac{s^2+1}{s(s+1)^2(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})^3(s+50)^2}|_{s=j1}$ *This one is easy.*
Partial Answer: $z = 0$. *Explain. Where are the zeros of this function? The function has zeros at* $s = \pm j1$. ■

Comment C.3.2 (The Importance of Complex Arithmetic)
Throughout the study of signals and systems, real sinusoids are routinely expressed in terms of complex exponentials. This follows from Euler's identities for cosine and sine:

$$\cos\omega_o t = Re\ e^{j\omega_o t} = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} \qquad \sin\omega_o t = Im\ e^{j\omega_o t} = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{j2} \qquad (C.97)$$

Given this, it is strongly recommended that beginning students become very proficient at complex arithmetic. This is the purpose of Exercise C.3.2. The exercise is also intended to convey basic approximation ideas that *will be exploited throughout our study of dynamical systems governed by linear ordinary differential equations with constant coefficients. See Section 3.5 (page 126) on sinusoidal steady state analysis.* ■

Exercise C.3.3 (Complex Arithmetic: A Prelude to Frequency Response Analysis)
In this exercise, we develop the required "complex arithmetic expertise" to analyze the so-called frequency response *of LTI systems (see Section 3.6 and Chapter 6). (1) For each of the following so-called* LTI system transfer functions H*, determine the so-called* magnitude response function $|H(j\omega)|$ *and so-called* phase response function $\angle H(j\omega)$*. (2) Sketch each of these functions versus* ω*. Use MATLAB to help you do the sketches. (3) For each case, determine approximate values for* $|H(j0.01)|$, $\angle H(j0.01)$, $|H(j100)|$, *and* $\angle H(j100)$.
Note: Use complex number (polar) pictures to help you derive the correct expression for the phase.
(a) $H(s) = \frac{1}{s}$, $|H(j\omega)| = \frac{1}{\omega}$, $\angle H(j\omega) = -90°$
(b) $H(s) = \frac{50}{s+10}$, $|H(j\omega)| = \frac{50}{\sqrt{\omega^2+100}}$, $\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{10}\right)$
(c) $H(s) = \frac{2}{s(s+2)}$, $|H(j\omega)| = \frac{2}{\omega\sqrt{\omega^2+4}}$, $\angle H(j\omega) = -90° - \tan^{-1}\left(\frac{\omega}{2}\right)$
(d) $H(s) = \left[\frac{100}{s+100}\right]^2$, $|H(j\omega)| = \frac{10^4}{\omega^2+10^4}$, $\angle H(j\omega) = -2\tan^{-1}\left(\frac{\omega}{100}\right)$
(e) $H(s) = \frac{1}{s}\left[\frac{100}{s+100}\right]^2$
(f) $H(s) = \frac{2}{s(s+2)}\left[\frac{100}{s+100}\right]^2$
(g) $H(s) = \frac{20}{s-5}$, $|H(j\omega)| = \frac{20}{\sqrt{\omega^2+25}}$, $\angle H(j\omega) = -\left[180° - \tan^{-1}\left(\frac{\omega}{5}\right)\right]$
(h) $H(s) = \frac{1}{s(s-1)}$
(i) $H(s) = \frac{3(s+\frac{2}{3})}{s(s-1)}$, $|H(j\omega)| = \frac{3\sqrt{\omega^2+(\frac{2}{3})^2}}{\omega\sqrt{\omega^2+1}}$, $\angle H(j\omega) = -90° - \left[180° - \tan^{-1}\left(\frac{\omega}{5}\right)\right] + \tan^{-1}\left(\frac{\omega}{\frac{2}{3}}\right)$
(j) $H(s) = \frac{3(s+\frac{2}{3})}{s(s-1)}\left[\frac{100}{s+100}\right]^2$
(k) $H(s) = \left[\frac{100-s}{100+s}\right]\frac{100}{s+100}$, $|H(j\omega)| = \frac{100}{\sqrt{\omega^2+10^4}}$, $\angle H(j\omega) = -3\tan^{-1}\left(\frac{\omega}{100}\right)$
(l) $H(s) = \frac{1}{s(s+1)}\left[\frac{100-s}{100+s}\right]\frac{100}{s+100}$
(m) $H(s) = \frac{1}{s^2+1}$, $|H(j\omega)| = \frac{1}{|1-\omega^2|}$, $\angle H(j\omega) = 0°$ *for* $\omega < 1$, $\angle H(j\omega) = -180°$ *for* $\omega > 1$, $\angle H(j1)$ *is indeterminate*
(n) $H(s) = \frac{1}{s^2+s+1}$, $|H(j\omega)| = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}$, $\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{1-\omega^2}\right)$ *for* $\omega \le 1$,
$\angle H(j\omega) = 180° - \tan^{-1}\left(\frac{\omega}{\omega^2-1}\right)$ *for* $\omega > 1$
(o) $H(s) = \frac{s}{s^2+s+1}$ ■

Comment C.3.3 (Right Triangles To Remember)
Exercise C.3.2 reminds us that it is useful to remember the trigonometric arithmetic associated with the following three (3) right triangles:

- *Angles:* 45°, 45°, 90° *Proportions:* 1, 1, $\sqrt{2}$;
- *Angles:* 60°, 30°, 90° *Proportions:* $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 1;
- *Angles:* 53°, 37°, 90° *Proportions:* 3, 4, 5. ■

Exercise C.3.4 (A Little Trigonometry Via Pictures: Use of Special Right Triangles)
Determine the sine, cosine, and tangent of each of the following angles by using pictures! (Use your calculator only as a check.)

(a) $\theta = 135°$	*(b)* $\theta = 120°$	*(c)* $\theta = 150°$	*(d)* $\theta = 127°$	*(e)* $\theta = 143°$
(f) $\theta = 225°$	*(g)* $\theta = 210°$	*(h)* $\theta = 240°$	*(i)* $\theta = 217°$	*(j)* $\theta = 233°$
(k) $\theta = 315°$	*(l)* $\theta = 300°$	*(m)* $\theta = 330°$	*(n)* $\theta = 307°$	*(o)* $\theta = 323°$

■

In analyzing dynamical systems, the problem of determining the roots of complex numbers arises often. This problem arises when computing the roots of polynomials. Such polynomial problems routinely arise when determining the natural modes or characteristic behaviors of a dynamical system. Given this, we now consider an example whose purpose is to develop proficiency at determining the roots of complex numbers.

Example C.3.1 (Roots of -1)

Two Roots of Unity. *Since the number* -1 *may be expressed as* e^{-j180° *and* e^{j180°, *it follows that the two roots which satisfy* $z^2 = -1$ *are given by*

$$z = e^{-j90^\circ},\ e^{j90^\circ}. \tag{C.98}$$

Three Roots of Unity. *Since the number* -1 *may be expressed as* e^{-j180°, e^{j180°, *and* e^{j540°, *it follows that the three roots which satisfy* $z^3 = -1$ *are given by*

$$z = e^{-j60^\circ},\ e^{j60^\circ},\ e^{j180^\circ}. \tag{C.99}$$

Four Roots of Unity. *Since the number* -1 *may be expressed as* e^{-j180°, e^{j180°, e^{j540°, *and* e^{-j540°, *it follows that the four roots which satisfy* $z^4 = -1$ *are given by*

$$z = e^{-j45^\circ},\ e^{-j45^\circ},\ e^{j135^\circ},\ e^{-j135^\circ}. \tag{C.100}$$

Five Roots of Unity. *Since the number* -1 *may be expressed as* e^{-j180°, e^{j180°, e^{j540°, e^{-j540°, *and* e^{j900°, *it follows that the five roots which satisfy* $z^5 = -1$ *are given by*

$$z = e^{-j36^\circ},\ e^{-j36^\circ},\ e^{j108^\circ},\ e^{-j108^\circ},\ e^{j180^\circ}. \tag{C.101}$$

Six Roots of Unity. *Since the number* -1 *may be expressed as* e^{-j180°, e^{j180°, e^{j540°, e^{-j540°, e^{j900°, *and* e^{-j900°, *it follows that the six roots which satisfy* $z^6 = -1$ *are given by*

$$z = e^{-j30^\circ},\ e^{-j30^\circ},\ e^{j90^\circ},\ e^{-j90^\circ},\ e^{j150^\circ},\ e^{-j150^\circ}. \tag{C.102}$$

■

Exponential Sinusoids From Complex Exponentials. The following example demonstrates how complex exponentials may be used to represent real exponential sinusoids. The example shows how cosine waveforms may be represented as the real part of a complex exponential waveform. Sine waveforms may be represented as the imaginary part of a complex exponential waveform.

In many applications, the real or imaginary prefix is intentionally dropped for notational economy. In such applications, one just works with the complex exponential waveform. Doing so, generally, is much easier than working with the original sinusoidal waveform.

Recall:

Exponentials are easier to multiply and differentiate than sinusoids.

Exponential solutions to differential equations are much nicer to work with than sinusoidal solutions.

After the analysis has been completed using the complex exponential waveform (e.g. determining a system's response to a complex exponential waveform), the real or imaginary prefix is reinserted to determine the signal (function) of interest - a real signal. This concept reappears throughout the study of signals, systems, controls, and most engineering disciplines. The concept fundamentally demonstrates the true utility of complex exponentials. We will see this in subsequent chapters.

Example C.3.2 (Generation of Exponential Sinusoids From Complex Exponentials)

One very important use of complex exponentials lies in their utility to represent real exponential sinusoids. This follows from the relationship:

$$\begin{aligned}
x(t) &= A\, e^{\sigma_o t}\cos(\omega_o t + \theta) && \text{(C.103)}\\
&= A\, e^{\sigma_o t}\, Re\{\, e^{j(\omega_o t+\theta)} \,\} && \text{(C.104)}\\
&= Re\{\, A\, e^{\sigma_o t}\, e^{j(\omega_o t+\theta)} \,\} && \text{(C.105)}\\
&= Re\{\, A e^{j\theta}\, e^{(\sigma_o + j\omega_o)t} \,\} && \text{(C.106)}\\
&= Re\{\, X\, e^{s_o t} \,\} && \text{(C.107)}
\end{aligned}$$

where

$$s_o = \sigma_o + j\omega_o \tag{C.108}$$

is called the complex frequency *of the exponential sinusoid and*

$$X \stackrel{\text{def}}{=} Ae^{j\theta} = |A|\, e^{j(\theta + \angle A)} \tag{C.109}$$

is called the phasor representation *or* complex representation *for the exponential-sinusoidal function* $x(\cdot)$.

The above relationship shows that exponential-cosines can be written as the real part of a complex exponential. (The above result is applied in Equations (3.138)-(3.140) in Section 3.5 on sinusoidal analysis of LTI systems.) Similarly,

$$\begin{aligned} x(t) &= A\, e^{\sigma_o t} \sin(\omega_o t + \theta) && \text{(C.110)} \\ &= A\, e^{\sigma_o t}\, Im\{\, e^{j(\omega_o t + \theta)}\, \} && \text{(C.111)} \\ &= Im\{\, A\, e^{\sigma_o t}\, e^{j(\omega_o t + \theta)}\, \} && \text{(C.112)} \\ &= Im\{\, Ae^{j\theta}\, e^{(\sigma_o + j\omega_o)t}\, \} && \text{(C.113)} \\ &= Im\{\, X\, e^{s_o t}\, \}. && \text{(C.114)} \end{aligned}$$

This relationship shows that exponential-sines can be written as the imaginary part of the complex exponential $X\, e^{s_o t}$. *We say that the above exponential sinusoids are generated from the complex exponential*

$$X\, e^{s_o t} = Ae^{j\theta}\, e^{(\sigma_o + j\omega_o)t} = A\, e^{\sigma_o t}\, e^{j(\omega_o t + \theta)} = |A|\, e^{\sigma_o t}\, e^{j(\omega_o t + \theta + \angle A)} \tag{C.115}$$

where $\angle A = 0$ *if* $A > 0$ *and* $\angle A = 180°$ *if* $A < 0$. *Note that the above has the form*

$$z(t) = |z(t)| e^{j\angle z(t)} = |A|\, e^{\sigma_o t}\, e^{j(\omega_o t + \theta + \angle A)} \tag{C.116}$$

where $|z(t)| = |A|\, e^{\sigma_o t}$ *and* $\angle z(t) = \omega_o t + \theta + \angle A$. *It thus follows that*

$$x(t) = Re\{\, z(t)\, \} = |A|\, e^{\sigma_o t} \cos(\omega_o t + \theta + \angle A). \tag{C.117}$$

and

$$x(t) = Im\{\, z(t)\, \} = |A|\, e^{\sigma_o t} \sin(\omega_o t + \theta + \angle A). \tag{C.118}$$

It is often very useful to associate the above real exponential sinusoids with their time-dependent phasor representations $Xe^{s_o t}$ *or their equivalent time-dependent complex representations* $z(t) = |A|\, e^{\sigma_o t}\, e^{j(\omega_o t + \theta + \angle A)}$. *Such association is very important in analyzing sinusoidally driven systems described by linear ordinary differential equations with constant coefficients. This association is made in many areas of science and engineering.* ∎

The following example demonstrates the utility of representing real sinusoids in terms of complex exponentials. The example specifically demonstrates how complex arithmetic may be used to greatly facilitate the addition of real sinusoids possessing the same frequency. The alternative, we know, is to use cumbersome trigonometric identities.

Example C.3.3 (Addition Of Sinusoids Using Complex Phasors)
Consider the continuous time signal

$$f(t) = 2\sin(\omega_o t - 30°) + \cos(\omega_o t + 45°). \tag{C.119}$$

This signal is the sum of two sinusoids of the same frequency ω_o. *The goal in this example is to illustrate how complex phasors may be used to add the sinusoids to get a single cosine signal of frequency* ω_o. *We proceed algebraically using the trigonometric relationship*

$$\sin(x) = \cos(x - 90^\circ) \tag{C.120}$$

and Euler's identities, as follows:

$$\begin{aligned} f(t) &= 2\cos(\omega_o t - 120^\circ) + \cos(\omega_o t + 45^\circ) = 2Re\{e^{j(\omega_o t - 120^\circ)}\} + Re\{e^{j(\omega_o t + 45^\circ)}\} & \text{(C.121)} \\ &= Re\{2e^{-j120^\circ} e^{j\omega_o t}\} + Re\{e^{j45\,\text{deg}} e^{j\omega_o t}\} = Re\{\left[2e^{-j120^\circ} + e^{j45^\circ}\right] e^{j\omega_o t}\} & \text{(C.122)} \\ &= Re\{X e^{j\omega_o t}\} = Re\{|X| e^{j(\omega_o t + \angle X)}\} = |X|\cos(\omega_o t + \angle X) & \text{(C.123)} \end{aligned}$$

where the critical calculation lies in the addition of complex phasors associated with each sinusoid as follows:

$$X = 2e^{-j120^\circ} + e^{j45^\circ} = -1 - j1.7321 + 0.7071 + j0.7071 = -0.2929 - j1.0249 = 1.0660e^{-j105.9481^\circ} \tag{C.124}$$

An Exercise. *For each of the following, show how you would determine the coefficients A_i and B_i using complex arithmetic:*

$$\begin{aligned} f_1(t) &= 10\sin(\omega_o t - 45^\circ) + \cos(\omega_o t + 135^\circ) = A_1\cos(\omega_o t + B_1) & \text{(C.125)} \\ f_2(t) &= -2\sin(\omega_o t + 60^\circ) + \cos(\omega_o t - 53^\circ) = A_2\sin(\omega_o t + B_2) & \text{(C.126)} \\ f_3(t) &= \sin(\omega_o t - 37^\circ) + 3\cos(\omega_o t + 30^\circ) = A_3\cos(\omega_o t + B_3). & \text{(C.127)} \end{aligned}$$

Specifically, you need to use trigonometric identities and complex exponential concepts to show that

$$\begin{aligned} A_1 e^{jB_1} &= 10e^{j(-45^\circ - 90^\circ)} + e^{j135^\circ} & \text{(C.128)} \\ A_2 e^{jB_2} &= -2e^{j60^\circ} + e^{j(-53^\circ + 90^\circ)} & \text{(C.129)} \\ A_3 e^{jB_3} &= e^{j(-37^\circ - 90^\circ)} + 3e^{j30^\circ}. & \text{(C.130)} \end{aligned}$$

You will need the following trigonometric identities:

$$\begin{aligned} \cos x &= \sin(x + 90^\circ) \quad (\cos\ \textit{leads}\ \sin\ \textit{by}\ 90^\circ) & \text{(C.131)} \\ \sin x &= \cos(x - 90^\circ) \quad (\sin\ \textit{lags}\ \cos\ \textit{by}\ 90^\circ) & \text{(C.132)} \end{aligned}$$

Addition Of Multiple Sinusoids. *The above shows that the addition of two sinusoids of the same frequency is equivalent to the addition of complex phasors. The same is true if we are adding many sinusoids of the same frequency. This is illustrated by the following algebraic steps:*

$$\begin{aligned} f(t) &= \sum_{k=1}^{n} |X_k|\cos(\omega_o t + \angle X_k) = \sum_{k=1}^{n} |X_k| Re\{e^{j(\omega_o t + \angle X_k)}\} = \sum_{k=1}^{n} Re\{|X_k| e^{j(\omega_o t + \angle X_k)}\} & \text{(C.133)} \\ &= \sum_{k=1}^{n} Re\{|X_k| e^{j\angle X_k} e^{j\omega_o t}\} = Re\{\sum_{k=1}^{n} |X_k| e^{j\angle X_k} e^{j\omega_o t}\} = Re\{\left[\sum_{k=1}^{n} |X_k| e^{j\angle X_k}\right] e^{j\omega_o t}\} & \text{(C.134)} \\ &= Re\{\left[\sum_{k=1}^{n} X_k\right] e^{j\omega_o t}\} = Re\{X e^{j\omega_o t}\} = Re\{|X| e^{j\angle X} e^{j\omega_o t}\} & \text{(C.135)} \\ &= Re\{|X| e^{j(\omega_o t + \angle X)}\} = |X|\cos(\omega_o t + \angle X) & \text{(C.136)} \end{aligned}$$

where

$$X = |X| e^{j\angle X} = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} |X_k| e^{j\angle X_k}. \tag{C.137}$$

It should be noted that it is just as easy to work with sines (instead of cosines) - just replace $\cos(\cdot)$ *with* $\sin(\cdot)$ *and replace all real parts (Re) with imaginary parts (Im).* ∎

Exercise C.3.5 (Addition of Sinusoids Via Phasors)
For each of the following, show how you would determine the coefficients A_i and B_i using complex arithmetic:

$$f_1(t) = -\sin(\omega_o t + 225°) + 4\cos(\omega_o t - 30°) - 2\sin(\omega_o t - 60°) = A_1\cos(\omega_o t + B_1) \tag{C.138}$$
$$f_2(t) = 4\cos(-\omega_o t + 150°) + 5\sin(\omega_o t + 53°) - \cos(\omega_o t - 30°) = A_2\sin(\omega_o t + B_2) \tag{C.139}$$
$$f_3(t) = -\sin(\omega_o t - 60°) - 10\cos(\omega_o t - 37°) + 2\sin(\omega_o t + 135°) = A_3\cos(\omega_o t + B_3). \tag{C.140}$$

Hint: Set up expressions for $A_i e^{jB_i}$.

The Big Motivator: Analysis of Dynamical Systems. The following example is intended to provide significant motivation for our implicit suggestion that if we are to analyze and design complex systems (e.g. integrated circuits, robots, airplanes, etc.), then we must eagerly embrace the world of complex numbers. While new ideas are presented in the example (perhaps prematurely), the punch line overwhelmingly justifies their introduction at this point! The results presented in the example are very profound - perhaps one of the most important results in the analysis of dynamical systems. The results are so profound that they arguably form the basis for 70% of all engineering analysis. (Some have even estimated that over 70% of all engineering courses would disappear if the results presented were not true. Imagine that!)

Example C.3.4 (Car Driven by Sinusoidal Force: Method of Transfer Function (MOTF))
This example illustrates how complex exponential may be used to study a dynamical system driven by a sinusoidal forcing function. We consider a car of mass $m = 1$. Let y denotes its speed. Two forces act on the car: a (rightward) force u due to the engine and an (leftward) aerodynamic force βy ($\beta = 1$).

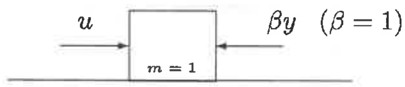

Figure C.5: Simple Model for Car ($m\dot{y} = u - \beta y$)

Model for Car Speed Dynamics. *Application of Newton's 2nd law shows that the speed y of the car is governed by the simple differential equation: $m\dot{y} = u - \beta y$ or (since $m = 1$ and $\beta = 1$)*

$$\dot{y} + y = u. \tag{C.141}$$

This differential equation can also be used to model many other physical systems; e.g. an RC-circuit with input voltage u and output voltage y.

New Concept: Transfer Function. *To make our case for complex exponentials, we need to introduce a new concept. Suppose we make the associations*

$$\frac{d}{dt} \longleftrightarrow s \qquad y \longleftrightarrow Y \qquad u \longleftrightarrow U. \tag{C.142}$$

(These associations will be formalized when we study Laplace Transforms. Do not fear.) With this association, we have

$$sY + Y = U. \tag{C.143}$$

Defining the ratio $H(s) \stackrel{\text{def}}{=} \frac{Y}{U}$ - called the system transfer function *- yields*

$$H(s) = \frac{1}{s+1}. \tag{C.144}$$

Now we consider a few simple questions - questions that might be asked in a differential equations class. Our approach to the questions is very significant because it forms the basis for most of the linear engineering analysis that is taught (to all engineering disciplines) around the world!

(a) **Exponential Input.** *Suppose that the applied input has the form $u(t) = Ae^{s_o t}$. We want to show that a particular solution to the system differential equation is $y_p(t) = H(s_o)Ae^{s_o t}$; i.e.*

$$\text{If } u(t) = Ae^{s_o t}, \text{ then } y_p(t) = H(s_o)\, Ae^{s_o t}. \tag{C.145}$$

To show this, we proceed as follows:

$$\dot{y}_p + y_p = s_o\, H(s_o)\, Ae^{s_o t} + A\, e^{s_o t} = (s_o + 1)\, H(s_o)\, A\, e^{s_o t} = A\, e^{s_o t} = u(t). \tag{C.146}$$

STOP! Please Take Note:

- Method of the Transfer Function. *What is significant about this result is that it applies to an arbitrary differential equation*

$$\frac{d^n y}{dt} + a_{n-1}\frac{d^{(n-1)}y}{dt} + \cdots + a_1\frac{dy}{dt} + a_o y = b_m\frac{d^m u}{dt} + b_{m-1}\frac{d^{(m-1)}u}{dt} + \cdots + b_1\frac{du}{dt} + b_o u \tag{C.147}$$

so long as $H(s_o)$ is well defined (not infinity) for the associated system transfer function:

$$H(s) \stackrel{\text{def}}{=} \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_o}{s^n + b_{n-1}s^{n-1} + \cdots + a_1 s + a_o} \tag{C.148}$$

Why is this? More generally, we have $\frac{d^k y_p}{dt} = s_o^k H(s_o)Ae^{s_o t}$ and $\frac{d^i u}{dt} = s_o^i Ae^{s_o t}$. Substituting y_p into the more general differential equation yields

$$\begin{aligned}
\frac{d^n y_p}{dt} &+ a_{n-1}\frac{d^{(n-1)}y_p}{dt} + \cdots + a_1\frac{dy_p}{dt} + a_o y_p && \text{(C.149)}\\
&= s_o^n H(s_o)Ae^{s_o t} + a_{n-1}s_o^{n-1}H(s_o)Ae^{s_o t} + \cdots + a_1 s_o H(s_o)Ae^{s_o t} + a_o H(s_o)Ae^{s_o t} && \text{(C.150)}\\
&= (s_o^n + a_{n-1}s_o^{n-1} + \cdots + a_1 s_o + a_o)\, H(s_o)\, Ae^{s_o t} && \text{(C.151)}\\
&= (s_o^m + b_{m-1}s_o^{m-1} + \cdots + b_1 s_o + b_o)\, Ae^{s_o t} && \text{(C.152)}\\
&= b_m\frac{d^m u}{dt} + b_{m-1}\frac{d^{(m-1)}u}{dt} + \cdots + b_1\frac{du}{dt} + b_o u. && \text{(C.153)}
\end{aligned}$$

This shows that our method for finding a particular solution - henceforth referred to as the

<u>Method of the Transfer Function</u>

applies whenever the system transfer function H is well defined at the frequency s_o of interest (i.e. $H(s_o)$ is a finite number, not infinity). This is very cool ... and it gets much better!

(b) **Complex Inputs.** *In general, the quantity A and the "frequency" s_o may be complex. (Be patient with me. I understand that real world signals are not complex valued. However, I also understand that real world signals may be written as the real or imaginary part of a complex signal! For example, $\cos\omega_o t = Re\,\{e^{j\omega_o t}\}$; $\sin\omega_o t = Im\,\{e^{j\omega_o t}\}$.) Given that A and s_o may be complex, show that*

$$\text{If } u(t) = Re\{\, Ae^{s_o t}\,\}, \text{ then } y_p(t) = Re\{\, H(s_o)\, Ae^{s_o t}\,\}. \tag{C.154}$$

$$\text{If } u(t) = Im\{\, Ae^{s_o t}\,\}, \text{ then } y_p(t) = Im\{\, H(s_o)\, Ae^{s_o t}\,\}. \tag{C.155}$$

Showing this amounts to simple bookkeeping if we define $u_r \stackrel{\text{def}}{=} Re\{\, Ae^{s_o t}\,\}$, $u_i \stackrel{\text{def}}{=} Im\{\, Ae^{s_o t}\,\}$, $y_{p_r} \stackrel{\text{def}}{=} Re\{\, H(s_o)\, Ae^{s_o t}\,\}$, $y_{p_i} \stackrel{\text{def}}{=} Im\{\, H(s_o)\, Ae^{s_o t}\,\}$. From (a) we know that if $u = u_r + ju_i$, then $y_p = y_{p_r} + jy_{p_i}$ is a particular solution; i.e.

$$\begin{aligned}
\dot{y}_p + y_p &= u && \text{(C.156)}\\
\frac{d}{dt}(y_{p_r} + jy_{p_i}) + (y_{p_r} + jy_{p_i}) &= u_r + ju_i && \text{(C.157)}\\
(\dot{y}_{p_r} + y_{p_r}) + j(\dot{y}_{p_i} + y_{p_i}) &= u_r + ju_i. && \text{(C.158)}
\end{aligned}$$

Equating the real and the imaginary parts on each side of the equation yields the following:

$$\dot{y}_{p_r} + y_{p_r} = u_r \qquad \dot{y}_{p_i} + y_{p_i} = u_i. \tag{C.159}$$

This corroborates the claim.
STOP! Please Take Note: In general, we can write:

$$A = |A|e^{j\angle A} \qquad H(s_o) = |H(s_o)|e^{j\angle H(s_o)} \qquad s_o = \sigma_o + j\omega_o. \tag{C.160}$$

Given this, it follows that

$$Ae^{s_ot} = |A|\ e^{j\angle A}\ e^{(\sigma_o+j\omega_o)t} = |A|\ e^{j\angle A}\ e^{\sigma_ot}\ e^{j\omega_ot} = |A|\ e^{\sigma_ot}\ e^{j(\omega_ot+\angle A)} \tag{C.161}$$

$$H(s_o)Ae^{s_ot} = |H(s_o)|\ e^{j\angle H(s_o)}\ |A|\ e^{j\angle A}\ e^{(\sigma_o+j\omega_o)t} \tag{C.162}$$

$$= |H(s_o)|\ e^{j\angle H(s_o)}\ |A|e^{j\angle A}\ e^{\sigma_ot}\ e^{j\omega_ot} = |A|\ |H(s_o)|\ e^{\sigma_ot}\ e^{j(\omega_ot+\angle A+\angle H(s_o))} \tag{C.163}$$

Given this, we have the following fundamental conclusions:

- **Constant Input.** *Consider a constant input. This corresponds to* $s_o = 0$ *(sometine referred to as a dc signal). Given this, we have the following*

  $$\text{If } u(t) = A, \text{ then } y_p(t) = AH(0). \tag{C.164}$$

  From this result, we see that a constant input results in a constant particular solution. Moreover, the input amplitude A *is multiplied by the factor* $H(0)$ *- the so-called* dc gain of the system.

- **Exponential Input.** *Suppose that* s_o *is real. We then have*

  $$\text{If } u(t) = Ae^{s_ot}, \text{ then } y_p(t) = H(s_o)\ Ae^{s_ot}. \tag{C.165}$$

  From this result, we see that an exponential input results in an exponential particular solution. The amplitude of the input exponential A *is multiplied by the factor* $H(s_o)$.

- **Sinusoidal Input.** *Suppose that* $s_o = j\omega_o$. *We then have*

  $$\text{If } u(t) = |A|\cos(\omega_ot + \angle A), \text{ then } y_p(t) = |A||H(j\omega_o)|\cos(\omega_ot + \angle A + \angle H(j\omega_o)). \tag{C.166}$$

  $$\text{If } u(t) = |A|\sin(\omega_ot + \angle A), \text{ then } y_p(t) = |A||H(j\omega_o)|\sin(\omega_ot + \angle A + \angle H(j\omega_o)). \tag{C.167}$$

  From this result, we see that a sinusoidal input results in a sinusoidal particular solution. Moreover, the input amplitude $|A|$ *is multiplied by the factor* $|H(j\omega_o)|$ *- the magnitude of the transfer function at* $s_o = j\omega_o$. *We also see that the system adds an additional phase shift to the input phase shift* $\angle A$*; the phase added is* $\angle H(j\omega_o)$ *- the angle (phase) of the system transfer function at* $s_o = j\omega_o$.

- **Exponential Sinusoidal Input.** *Suppose that* $s_o = \sigma_o + j\omega_o$. *We then have*

  $$\text{If } u(t) = |A|e^{\sigma_ot}\cos(\omega_ot + \angle A), \text{ then } y_p(t) = |A||H(s_o)|e^{\sigma_ot}\cos(\omega_ot + \angle A + \angle H(s_o)). \tag{C.168}$$

  $$\text{If } u(t) = |A|e^{\sigma_ot}\sin(\omega_ot + \angle A), \text{ then } y_p(t) = |A||H(s_o)|e^{\sigma_ot}\sin(\omega_ot + \angle A + \angle H(s_o)). \tag{C.169}$$

  From this result, we see that an exponential sinusoidal input results in an exponential sinusoidal particular solution. Moreover, the input amplitude $|A|$ *is multiplied by the factor* $|H(s_o)|$ *- the magnitude of the transfer function at* $s_o = \sigma_o + j\omega_o$. *We also see that the system adds an additional phase shift to the input phase shift* $\angle A$*; the phase added is* $\angle H(s_o)$ *- the angle (phase) of the system transfer function at* $s_o = \sigma_o + j\omega_o$.

STOP! Take Note: Why is the above significant? Here are some reasons.

- **Wide Applicability of Method of the Transfer Function.** *Many dynamical systems and processes may be approximately modeled by linear ordinary differential equations with constant coefficients. The Method of the Transfer Function presented above applies to such systems and to any system possessing a so-called transfer function. Many nonlinear systems, for example, may be linearized; e.g. the nonlinear (grandfather clock) pendulum equation*

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{1}{ml^2}u, \tag{C.170}$$

for small θ, may be approximated by the linear system:

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{1}{ml^2}u. \tag{C.171}$$

This linear ordinary differential equation (LTI system; see Chapter 3) has transfer function

$$H(s) = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}. \tag{C.172}$$

This shows the wide applicability of the method of the transfer function (MOTF).

- **Traditional Method Stinks!** *One way to quickly appreciate the significance of the above results is to recall the method taught in traditional differential equation texts. For years, the texts have "pushed" the* Method of Undetermined Coefficients. *If the differential equation is low order (one, two), then the traditional method is ok. One guesses a standard form for the particular solution, differentiates a few times, substitutes into the differential equation, and finds the "undetermined coefficients." If the order of the differential equation is high (say 4 and above), then the method of undetermined coefficients becomes the METHOD OF MANY PAGES! For any real problem involving a high order differential equation (e.g. 12th order for any 6 degree-of-freedom jet), then one would not even think of using the traditional method. While the traditional method becomes impossible to attempt, our so-called method of the transfer function permits us to write down a particular solution immediately! Moreover, the only cumbersome coefficient calculation that we must perform is simple and involves the system transfer function. Since the transfer function in general involves the system's parameters (e.g. masses, resistances, capacitances, etc.), it follows that the calculation that we perform (involving H) can provide significant insight into how the particular solution depends on the system parameters.*

- **Many Different Inputs are Accommodated by Method.** *The method presented permits us to handle constants, exponentials, sinusoids, and exponential sinusoids. Sinusoids are by far the most important signals in engineering. They are used everywhere as test signals. They are used as test signals for amplifiers, aircraft, robots, engines, the brain, chemical systems, etc.*

  When one purchases an audio amplifier, they would like it to "behave linearly over a wide frequency range." What does this mean? This means that sinusoidal signals ranging in frequency from 0 to 20 kHz should be amplified uniformly without distortion. This requirement is one that most people take for granted. Similar requirements, are essential for many systems to perform properly. A missile flight control system, for example, should be designed to follow the acceleration commands that are issued by the on board guidance system. Such signals may vary over a very wide range of frequencies.

- **Useful for Steady State Analysis.** *When an input is applied to a system, the input excites the natural modes (tendencies) of the system; i.e. the system does what it naturally wants to do. If these natural modes are stable (i.e. decay with time), then we say that the system is stable. In such a case our method of the transfer function not only gives us a particular solution, it provides a steady state solution. We note that this solution is the steady state output of the system.* ∎

The method of the transfer function (MOTF) presented above will be examined in several chapters (Section 3.5, Section 4.8.5, Chapter 6, Appendix A.1). It is, once again, one of the most important results in the analysis of dynamical systems. Because of this method, linear time invariant systems (e.g.. systems governed by linear ordinary differential equations with constant coefficient) are nice to analyze. For nonlinear systems, no analogous method exists! (Unless the system is "almost linear.") This explains why nonlinear systems are so difficult to analyze. It also explains why engineers often work very hard to build systems that exhibit linear behavior!

C.4 Summary and Conclusions

In this chapter, we provided an overview of fundamental results from different areas of mathematics.